Epsilon–Substitution Method for the Ramified Language and Δ^1_1-Comprehension Rule

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Abstract

We extend to Ramified Analysis the definition and termination proof of Hilbert's ϵ-substitution method. This forms a base for future extensions to predicatively reducible subsystems of analysis. First such system treated here is second order arithmetic with Δ^1_1-comprehension rule.

Introduction

The epsilon substitution method is based on the language introduced by Hilbert [7]. The main non-boolean construction of this language is $\epsilon x F[x]$, read as "an x satisfying the condition $F[x]$". When x is numerical, it is interpreted as the least x satisfying $F[x]$". Existential and universal quantifiers become explicitly definable by $\exists x F[x] := F[\epsilon x F[x]]$ and $\forall x F[x] := F[\epsilon x \neg F[x]]$.

The main axioms of the corresponding formalism are *critical formulas*

$$F[t] \to F[\epsilon x F[x]] \tag{1}$$

Hilbert suggested an approach (Ansatz) to transforming arbitrary (non-finitistic) number-theoretic proofs into finitistic (combinatorial) proofs by means of the substitution method using substitution S of numerals for (a finite number of) constant epsilon-terms (cf. [7, 15]). If all critical formulas (1) are true under S, it is called a solving substitution for the system E.

Hilbert's approach is based on the following idea of generating substitutions of numerals for closed epsilon-terms. The initial approximation S_0 is identically 0. If substitutions $S_0, \ldots, S_i$ are already generated, and S_i is not yet a solving substitution, then S_{i+1} is found by putting

$$S_{i+1}(\epsilon x F) = (\textit{the least } N \leq t)\ (S_i(F[n]) = \textit{true})$$

for the first formula which is false under S_i.

The problem stated by Hilbert was to prove termination of the sequence $S_0, S_1, S_2, \ldots$ after a finite number of steps for any system of critical formulas (1). After von Neumann's [17] attack on this (see below) Hilbert [6] stated further problems:

find a proof of termination for pure number theory, for analysis, and for analysis with the axiom of choice, when each of these systems is suitably reformulated in the epsilon-calculus.

In this paper we extend to Ramified Analysis the definition and termination proof of the ϵ-substitution method given in [15]. This forms a base for future extensions to predicatively reducible subsystems of analysis.

In the present paper we consider a subsystem DCR of second order arithmetic based on the Δ_1^1-comprehension rule:

$$\frac{\forall x(\exists Y A \leftrightarrow \forall Z B)}{F[\lambda x \exists Y A] \rightarrow \exists X F}$$

A, B being arithmetic formulas. This system is predicative and can be embedded into ramified analysis of level less than ω^ω. However, the embedding presented in [19] uses an infinitary system, and simulating it in our framework would lead to an infinite system of critical ϵ-formulas. We use a modification which results in a finite system of critical ϵ-formulas having a variable stratification (assignment of levels to second order variables) which depends on a current ϵ-substitution.

The definition of ϵ-substitution for DCR is the same as for ramified analysis, but the coordination of variable ordinal assignments with pure RA-aspects required additional work. The termination proof for RA is an extension of the non-effective termination proof from [15].

Cf. [15] for more motivation and history.

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1 The System $RA\epsilon$

The formulation here is close to [7] and extends [15]. Instead of usual computable functions we use computable predicates (their graphs) to simplify technical details.

1.1 The language $LA\epsilon$

Basic Symbols

0-variables (denoted by $x, y, z, \ldots$);

1-variables (denoted by $X, Y, Z, \ldots$);

the 0-ary function constant 0 (zero), and the unary function constant S (successor);

predicate constants for n-ary computable predicates ($n \geq 1$) including = (equality), *add* (addition) and *prod* (multiplication);

the propositional logical connectives $\neg$, $\wedge$, $\rightarrow$;

the epsilon symbol ϵ and the application symbol App.

Each 1-variable X is assigned a level $lev(X) \in On - \{0\}$. Sometimes X is written as X^α where $\alpha = lev(X)$.

Definition 1 (Terms and formulas)

1. Each ι-variable is a ι-term ($\iota = 0, 1$).

2. *The constant 0 is a 0-term.*
3. *If t is a 0-term, then $\mathsf{S}t$ is a 0-term.*
4. *If t_1,...,t_n are 0-terms and p is an n-ary predicate constant, then $pt_1 \dots t_n$ is a formula.*
5. *If t is a 0-term and P is a 1-term then $\mathsf{App}Pt$ is a formula.*
6. *If A, B are formulas then $\neg A$, $\wedge AB$, $\rightarrow AB$ are formulas.*
7. *If F is a formula and ξ is a ι-variable then $\epsilon\xi F$ is a ι-term (ι=0,1).*

To increase readability, we sometimes use infix notation for binary logical connectives, insert parentheses and use standard abbreviations like $A \vee B = \neg A \rightarrow B$.

Definition 2 (λ-terms)
If G is a formula and z is a 0-variable occurring free in G, then λzG is a λ-term.

ι-terms ($\iota = 0, 1$), formulas and λ-terms are called *expressions*.

Note that lambda-terms are not allowed to occur inside other expressions. The result of their substitution into other expressions is understood via λ-conversion (cf. Definition 9).

Terms of the form $\epsilon\xi F$ are called *ϵ-terms*.

The 0-terms $0, \mathsf{S}0, \mathsf{SS}0, ...$ are called *numerals*.

Var denotes the set of all variables and $\mathbb{N}$ denotes the set of all numerals.

We define $0^0 := 0, \quad 0^1 := \lambda z(z = 0)$.

For each term u we set

$$\iota(u) := \begin{cases} 0 & \text{if } u \text{ is a 0-term} \\ 1 & \text{if } u \text{ is a 1-term} \end{cases}$$

.

Syntactic variables:
e, u, v, w for expressions,
ξ, η for variables,
p for any predicate constant and the symbols $\mathsf{S}, \neg, \wedge, \rightarrow, \mathsf{App}$,
s, t for 0-terms,
n for a numeral $\mathsf{S}^n 0$,
P for 1-terms,
T for 1-terms and λ-terms,
A, B, F, G for formulas,
α, β, γ for ordinals

The sets $FV(e)$, $BV(e)$ of free and bound variables of an expression e are defined in the standard way. An expression e is called closed iff $FV(e) = \emptyset$.

Substitution
We identify expressions which are equivalent modulo renaming of bound variables. If u is a $\iota(\xi)$-term then $e[\xi/u]$ denotes the result of substituting u for each free occurrence of ξ in e, where bound variables in e are renamed if necessary. If ξ is known from the context we write $e[u]$ for $e[\xi/u]$.

Recall that the level of a formula of ramified analysis (language RA below) is the maximum level of predicate variables in it. We define level for $RA\epsilon$ in such a way that it is invariant under substitution of 1-terms for variables: $X^\alpha/\epsilon X^\alpha F$. More formally, we define below by induction on e the value $lev_{\mathbf{x}}(e)$ where variables in the list $\mathbf{x} = \xi_1, \dots \xi_n$ are treated as bound, and set

$$lev(e) = lev_{\emptyset}(e)$$

It is assumed that the bound variables in e are renamed so that no variable occurs both free and bound, no variable is bound twice, and no bound variable of e occurs in $\mathbf{x}$.

Definition 3 *1.* $lev_{\mathbf{x}}(0) = 0$,

$lev_{\mathbf{x}}(\xi) := \begin{cases} lev(\xi) & \text{if } \xi \notin \mathbf{x} \\ 0 & \text{otherwise} \end{cases}$

2. $lev_{\mathbf{x}}(pe_1 \dots e_n) = \max_i\{lev(e_i)\}$

3. $lev_{\mathbf{x}}(\epsilon\xi F) := \begin{cases} lev(\xi) & \text{if } \mathbf{x} \cap FV(\epsilon\xi F) = \emptyset \\ \max\{lev(\xi), lev_{\mathbf{x}\xi}(F)\} & \text{otherwise} \end{cases}$

4. $lev_{\mathbf{x}}(\lambda z G) = lev_{\mathbf{x},z}(G)$

Definition 4 *An epsilon-term $\epsilon\xi F$ or a λ-term $\lambda z F$ is* canonical *if it is closed and F does not contain closed ϵ-subterms.*

TRUE *(*FALSE*) denotes the set of all true (false) simple formulas.*

$\mathbb{B}_0 := \mathbb{N}$,

$\mathbb{B}_\sigma :=$ the set of all canonical λ-terms e with level $lev(e) < \sigma$ for $\sigma \neq 0$.

Lemma 1 *Let $\mathbf{x} \cap BV(e) = \emptyset$, $\mathbf{x} \supseteq FV(e)$ and $FV(u) \neq \emptyset$ for any ϵ-subterm u of e. Then* $lev_{\mathbf{x}}(e) = \max\{\alpha : X^\alpha \in BV(e)\}$

In particular, if $\lambda z G$ is canonical, then $lev_{\mathbf{x}}(\lambda x G) = \max\{\alpha : X^\alpha \in BV(\lambda z G)\}$

Proof: induction on e.

1. $e = \xi, 0$. Trivial.

2,4. $e = pe_1, \dots, e_n$ or $\lambda z G$. Use I.H.

3. $e = \epsilon\xi F$. Since $\mathbf{x} \supseteq FV(\epsilon\xi F) \neq \emptyset$, one has $lev_{\mathbf{x}}(e) = \max\{lev(\xi), lev_{\mathbf{x}\xi}(F)\}$. Use I.H. □

Lemma 2 *Let $\lambda z G$ be a canonical λ-term, and $G[z/0]$ contain a closed subterm $\epsilon X^\alpha F$. Then $lev(\lambda z G) \geq \alpha$.*

Proof. One has $\epsilon X^\alpha F = \epsilon X^\alpha H[z/0]$ with $FV(\epsilon X^\alpha H) = \{z\}$. By the Lemma 1, $lev(\lambda z G) \geq \alpha$. □

Definition 5 Substitution $e[Y/T]$ *for $t = \lambda z G$ is the result of replacing* AppTt *by $G[t]$.*

Definition 6 *Substitutable terms.*

Let $\alpha \in On, \alpha > 0$. Then T is α- substitutable *if T is a 1-term and $lev(T) \leq \alpha$, or T is a λ-term and $lev(T) < \alpha$.*

0-substitutable *terms are exactly 0-terms.*

Lemma 3 *If $u = \epsilon\eta A$ is a term, $\xi \notin \mathbf{x}$, $\mathbf{x} \cap FV(u) = \emptyset$ and $lev(\eta) = lev(\xi)$ then $lev_{\mathbf{x}}(e[\xi/u]) = lev_{\mathbf{x}}(e)$; in particular,*

$lev(\lambda z G[\xi/u]) = lev(\lambda z G)$.

Proof: easy induction on e. □

1.2 Axioms and inference rules of $RA\epsilon$

The language of $RA\epsilon$ is $LA\epsilon$.

The only **inference rule** of $RA\epsilon$ is *modus ponens*: $\frac{F \quad F \to G}{G}$.

Axioms of $RA\epsilon$

Propositional axioms: all propositional tautologies of the language $LA\epsilon$,

All substitution instances of defining axioms for the predicate constants, including the predicates of addition and multiplication:
$add(s, 0, s)$ and $add(s, t, r) \to add(s, \mathsf{S}t, \mathsf{S}r)$
$prod(s, 0, 0)$ and $prod(s, t, r) \wedge add(r, s, r_1) \to prod(s, \mathsf{S}t, r_1)$

Equality axioms: $t = t$ and $s = t \to (F[s] \to F[t])$,

Peano axioms for S: $\mathsf{S}t \neq 0$ and $\mathsf{S}s = \mathsf{S}t \to s = t$,

Minimality axioms: $\epsilon x F[x] = \mathsf{S}t \to \neg F[t]$,

Critical formulas:
$F[t] \to F[\epsilon x F[x]]$, (first order critical formulas)
$s \neq 0 \to F[\epsilon x F[x]]$ with $F := (s = \mathsf{S}x)$ i.e. $s \neq 0 \to s = \mathsf{S}\epsilon x(s = \mathsf{S}x)$,
$F[T] \to F[\epsilon X^\alpha F[X^\alpha]]$ (second order critical formulas)
where T is α-substitutable.

Comment. Critical formulas of the second kind are not present in [7]. They are needed here to interprete Robinson's axiom $s \neq 0 \to \exists x(s = \mathsf{S}x)$.

This concludes the description of $RA\epsilon$. Note that the formulas $s = t \to (\epsilon x F[x, s] = \epsilon x F[x, t])$ are consequences of the equality axioms of $RA\epsilon$. Formulas $\forall x(F \leftrightarrow G) \to \epsilon x F = \epsilon x G$ are not axioms (and in general are not derivable) but can be easily included (cf. [12]).

Lemma 4 *$RA\epsilon$ is closed under the* substitution rule*: if F is derivable, $u = \epsilon\eta H$ and $lev(\xi) = lev(\eta)$, then $F[\xi/u]$ is derivable.*

Proof: All axioms and inference rules of $RA\epsilon$ are closed under substitution. In particular, a second-order critical formula $F[T] \to F[\epsilon X^\alpha F[X^\alpha]]$ is transformed into

$$F[u, T[u]] \to F[u, \epsilon X^\alpha F[u, X^\alpha]]$$

and $lev(T[u]) = lev(T)$ by the Lemma 3.

2 Embedding RA into $RA\epsilon$

RA is a formal system of *ramified analysis* (i.e. second order arithmetic with the axiom scheme of comprehension for stratified formulas).

The language LA of RA is obtained from $LA\epsilon$ by dropping the epsilon symbol ϵ and adding the existential quantifier $\exists$. In the definition of terms and formulas case 7 is replaced by:

7. If F is a formula and ξ is a ι-variable ($\iota = 0, 1$) then $\exists\xi F$ is a formula.

Note that the only 1-terms of LA are 1-variables. λ-terms are expressions of the form $\lambda z G$ where G is a formula. They are treated as in $RA\epsilon$.

A it level of an expression of RA is the maximum level of variables in it.

The universal quantifier is defined via $\exists$: $\forall\xi F := \neg\exists\xi\neg F$.

Axioms of RA

(1) Propositional axioms: all propositional tautologies of the language LA,

(2) Defining axioms for the predicate constants, including the predicates of addition and multiplication:

$add(x, 0, x)$ and $add(x, y, z) \to add(x, \mathsf{S}y, \mathsf{S}z)$

$prod(x, 0, 0)$ and $prod(x, y, z) \wedge add(z, x, z_1) \to prod(x, \mathsf{S}y, z_1)$

(3) Equality axioms: $x = x$ and $x = y \to (F[x] \to F[y])$,

(4) Peano axioms for S: $\mathsf{S}x \neq 0$ and $\mathsf{S}x = \mathsf{S}y \to x = y$,

(5) Induction axioms: $F[0] \to \forall x(F[x] \to F[\mathsf{S}x]) \to \forall x F[x]$,

(6) First order existential axioms: $F[t] \to \exists x F[x]$,

(7) Second order existential axioms: $F[T] \to \exists X F[X]$,

where T is a 1-variable of a level $\leq \alpha$ or a λ-term of a level $< \alpha$.

Inference rules of RA

$$\text{modus ponens} \quad \frac{F \quad F \to G}{G}$$

$$\exists\to \quad \frac{F[\eta] \to G}{\exists\xi F[\xi] \to G}$$

where ξ, η are both 0-variables or 1-variables

of the same level, and the standard proviso is satisfied: the *eigenvariable* η does not occur free in the conclusion $\exists\xi F[\xi] \to G$.

Definition 7

For any formula F of EA define inductively an $LA\epsilon$-formula F^*:

$F^* := F$ for atomic F

$(\neg F)^* := \neg F^*$ and $(F \odot G)^* := (F^* \odot G^*)$ for $\odot = \wedge, \to$

$(\exists\xi F[\xi])^* := F^*[\epsilon\xi F[\xi]^*]$

$(\lambda z G)^* := \lambda z G^*$

Theorem 1

a) If $EA \vdash F$ then $EA\epsilon \vdash F^*$.

b) If $EA \vdash F$ and F is closed then there exists an $EA\epsilon$-derivation of F^* in which all formulas are closed.

Proof: standard, using the fact that the *-translation of any existential axiom of EA is a critical formula. $\Box$

3 Computations with the ϵ-Substitutions

Definition 8

An ϵ-substitution *is a function S such that*

$dom(S)$ (domain of S) is a set of canonical ϵ-terms,

if $e \in dom(S)$ then $S(e) \in \mathbb{B}_{lev(e)} \cup \{?\}$.

An ϵ-substitution S is called total *if $dom(S)$ is the set of all canonical ϵ-terms.*
$\overline{S} := S \cup \{(e, ?) : e$ *is a canonical ϵ-term* $\notin dom(S)\}$ *is called the* standard extension of S.

3.1 Computation Steps

Let S be an arbitrary ϵ-substitution.

Definition 9 (Inductive definition of $e \hookrightarrow^1_S e'$)
1.1. If $(e, u) \in S$ and $u \neq ?$ then $e \hookrightarrow^1_S u$
1.2. If $(e, ?) \in S$ then $e \hookrightarrow^1_S 0^{\iota(e)}$
2. If $1 \leq i \leq n$, $e_i \hookrightarrow^1_S e'_i$ and e'_i is not a λ-term then $pe_1 \dots e_n \hookrightarrow^1_S pe_1 \dots e_{i-1}e'_ie_{i+1} \dots e_n$
3. If $(P, \lambda zG) \in S$ then $\mathsf{App}Pt \hookrightarrow^1_S G[t]$
4. If $F \hookrightarrow^1_S F'$ then $\epsilon\xi F \hookrightarrow^1_S \epsilon\xi F'$
5. If $G \hookrightarrow^1_S G'$ then $\lambda zG \hookrightarrow^1_S \lambda zG'$.

Definition 10
*e is called S-*reducible *if there exists an e' with $e \hookrightarrow^1_S e'$. Otherwise e is called S-*irreducible *or* in S-normal form. *$\hookrightarrow_S$ denotes the transitive and reflexive closure of $\hookrightarrow^1_S$.*

Lemma 5 *If $e \hookrightarrow^1_S e'$ then $FV(e') = FV(e)$.*

Proof is standard. □

We are going to prove that $\hookrightarrow_S$ is well-founded by establishing that the value $d(e)$ defined below decreases during computations. The depth $d(e)$ is a measure of nesting of closed ϵ-subterms, taking into account that 1-ϵ-subterms can be substituted by λ-terms, and this can increase nesting by an arbitrary finite amount. It uses the natural sum function # on ordinal numbers, which is a commutative associative analog of the ordinal sum +:

$$\omega^\alpha \# \omega^\beta = \omega^{max(\alpha,\beta)} + \omega^{min(\alpha,\beta)},$$

and the function:

$$\omega_2(\alpha) := \begin{cases} \omega^{\omega^{\alpha-1}} & \text{if } 0 < \alpha < \omega \\ \omega^{\omega^\alpha} & \text{otherwise} \end{cases}$$

Definition 11
1. $d(\xi) := d(0) := 0$
2. $d(pe_1 \dots e_n) := d(e_1)\# \dots \#d(e_n)$, for $p \neq \mathsf{App}$
3. $d(\mathsf{App}Pt) := \begin{cases} d(P)\ \#d(t) \cdot (\omega_2(\alpha) + 1)\ \#\omega_2(\alpha) & \text{if } P = \epsilon X^\alpha F \textit{ is closed} \\ d(P)\#d(t) & \text{otherwise} \end{cases}$
4. $d(\epsilon\xi F) := \begin{cases} d(F) + 1 & \textit{if } \epsilon\xi F \textit{ is closed} \\ d(F) & \textit{otherwise} \end{cases}$
5. $d(\lambda zG) := d(G)$

Note. $d(e) = 0$ iff e has no closed ϵ-subterms.

An ϵ-term $e = \epsilon\xi F$ is canonical (cf. Definition 4) if it is closed and $d(e) = 1$ (i.e. $d(F) = 0$). A λ-term λzG is canonical if $FV(G) = \{z\}$ and $d(G) = 0$.

Definition 12 *1. e is called* simple *if $d(e) = 0$ and e is closed.*

2. TRUE (FALSE) *denotes the set of all true (false) simple formulas. [Note that a simple formula contains no variables and is constructed from computable atomic formulas by boolean connectives. Every simple term is a numeral].*

Comment. The objects to be immediately evaluated are canonical ϵ-terms, and the values of the terms of level α will be elements of B_α (cf. the Definition of an ϵ-substitution below in the Section 3).

For uniformity we extend the 'operation' App:

Definition 13

$$\mathsf{App}(T,t) := \begin{cases} G[t] & \text{if } T = \lambda z G \\ \mathsf{App}Tt & \text{otherwise} \end{cases}$$

For $p \neq \mathsf{App}$ we set $p(e_1, \ldots, e_n) := pe_1 \ldots e_n$.

Let us recall some necessary properties of ordinal functions.

Lemma 6 *(a) Ordinals of the form ω^γ are additive principal: $\alpha, \beta < \omega^\gamma \to \alpha\#\beta < \omega^\gamma$.*

(b) Ordinals of the form ω^{ω^γ} are multiplicative principal: $\alpha, \beta < \omega^{\omega^\gamma} \to \alpha \cdot \beta < \omega^{\omega^\gamma}$.

(c) Functions $\#$, $\lambda\alpha\omega_2(\alpha)$ are strictly monotone.

(d) The function $\lambda\alpha\ \alpha \cdot (\beta + 1)$ is strictly monotone.

Proof. For (a)-(c) cf. [18].

(d) If $\alpha < \alpha'$ then $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha \leq \alpha' \cdot \beta + \alpha < \alpha' \cdot \beta + \alpha' = \alpha' \cdot (\beta + 1)$. □

Definition 14 *$lev^+(e) := \sup\{\alpha : e$ contains a closed subterm $\epsilon X^\alpha F\}$*

Lemma 7 *a) If $lev^+(e) < \beta$ then $d(e) < \omega_2(\beta)$,*
b) If $lev^+(e[y/0]) < \beta$ then $d(e[y/t]) < (d(t) + 1) \cdot \omega_2(\beta)$.

Proof: induction on e.

a) Consider first the case when $e = \mathsf{App}Pt$ with a closed $P = \epsilon X^\alpha F$. Then $\alpha < \beta$, $lev^+(P) < \beta$, $lev^+(t) < \beta$. By the I.H. $d(P), d(t) < \omega_2(\beta)$, and by the Lemma 6 $d(e) = d(P)\#d(t) \cdot (\omega_2(\alpha) + 1)\#\omega_2(\alpha) < \omega_2(\beta)$.

All remaining cases are trivial or handled by I.H. and the Lemma 6(a).

b) Let $y \in FV(e)$, since otherwise $d(e[y/t]) = d(e) < \omega_2(\beta)$. Set $\gamma := (d(t) + 1) \cdot \omega_2(\beta)$ by (a).

1. If $e = y$ then $d(e[y/t]) = d(t) < d(t) + 1 < \gamma$.

2. If $e = pe_1 \ldots e_n$ with $d(e[t]) = d(pe_1[t] \ldots e_n[t]) = d(e_1[t])\# \ldots \#d(e_n[t])$ then by the induction hypothesis (I.H.) $d(e_i[t]) < \gamma$ and $d(e[t]) < \gamma$ since γ is closed under $\#$.

3. Let $e = \mathsf{App}Ps$ with $P = \epsilon X^\alpha F$ such that $P[t] = \epsilon X^\alpha F[t]$ is closed. Then $\epsilon X^\alpha F[0]$ is closed, $\alpha \leq lev^+(e[0]) < \beta$, $lev^+(s[0]) \leq lev^+(e[0]) < \beta$, and by the I.H. $d(P[t]), d(s[t]) < \gamma$, hence by the Lemma 6(b) $d(s[t]) \cdot \omega_2(\alpha) < \gamma$ and by the Lemma 6(a) $d(e[t]) = d(P[t])\#d(s[t]) \cdot (\omega_2(\alpha) + 1)\#\omega_2(\alpha) < \gamma$.

4. If $e = \epsilon\xi F$ then we can assume $\xi \notin FV(t)$ and $d(e[t]) = d(\epsilon\xi F[t]) \leq d(F[t]) + 1 \overset{IH}{<} \gamma$.

5. If $e = \lambda z F$, apply I.H. □

Lemma 8 *If* $e \hookrightarrow_S^1 e'$ *then* $d(e') < d(e)$.

Proof by induction on the definition of $\hookrightarrow_S^1$.

1.1, 1.2. If e is a canonical ϵ-term then $d(e') = 0 < 1 = d(e)$.

2. If $e = pe_1 \dots e_n,\ e' = pe_1 \dots e_i' \dots e_n$ and $e_i \hookrightarrow_S^1 e_i'$, then:

2.1. if $p \neq \mathsf{App}$ then $d(e') = d(e_1)\# \dots \#d(e_i')\# \dots \#d(e_n) \overset{IH}{<} d(e_1)\# \dots \#d(e_n) = d(e)$.
2.2. if $e = \mathsf{App}Pt$ and $e' = \mathsf{App}P't'$, then $d(e) = d(P)\#d(t) \cdot (\omega_2(\alpha) + 1)\#\omega_2(\alpha)$,
$d(e') = d(P')\#d(t') \cdot (\omega_2(\alpha) + 1)\#\omega_2(\alpha)$, and we use I.H. and the Lemma 6 (c),(d).
3. Let $e = \mathsf{App}Pt$ with a closed $P = \epsilon X^\beta F$ and $e' = G[t]$ with $lev(\lambda zG) < \beta$. Then by the lemma 2, $lev^+(G[0]) \leq lev(G) < \beta$. Hence by Lemma 7(b) $d(e') < (d(t) + 1) \cdot \omega_2(\beta)$, and
$d(e) = d(P)\#d(t) \cdot (\omega_2(\beta) + 1)\#\omega_2(\beta) = d(P)\#(d(t) \cdot \omega_2(\beta) + d(t))\#\omega_2(\beta) \geq d(t) \cdot \omega_2(\beta)\#\omega_2(\beta)$
$= (d(t) + 1) \cdot \omega_2(\beta) > d(e')$

4, 5. If $e = \epsilon\xi F, e' = \epsilon\xi F'$ or $e = \lambda zF, e' = \lambda zF'$ with $F \hookrightarrow_S^1 F'$ then, since $FV(F) = FV(F')$, we have $d(e') = d(F') + j \overset{IH}{<} d(F) + j = d(e)$ for some $j = 0, 1$. $\square$

Theorem 2 (Church-Rosser Property)
For each expression e *there exists a unique* S*-irreducible expression* e^* *with* $e \hookrightarrow_S e^*$.

Proof. By Lemma 8 computations terminate, and it is proved in a standard way that the relation $\hookrightarrow_S^1$ is locally confluent. $\square$

Definition 15 (Normal form $|e|_S$ for expression e)
The unique expression e^* *in the previous theorem is called* the S-normalform of e *and denoted by* $|e|_S$.

Lemma 9 *The relation* $\hookrightarrow_S$ *is preserved under substitution.*
a) $e \hookrightarrow_S^1 e'$ *implies* $e[\eta/u] \hookrightarrow_S e'[\eta/u]$.
b) $u \hookrightarrow_S^1 u'$ *implies* $e[\eta/u] \hookrightarrow_S e[\eta/u']$.
c) $e[\eta/u] \hookrightarrow_S |e|_S[\eta/|u|_S]$.

Proof: standard, cf. [15], section 3.

4 The rank function

The rank will measure nesting of bound variables. We extend to the ϵ-language a definition known for Ramified Analysis [19].

Set $o(0) := o(x) := 0$, $o(X^\alpha) := \omega \cdot \alpha$.

$$o(e) = \sup\{o(\xi) : \xi \text{ occurs in } e\}$$

In the following σ denotes elements from $\mathrm{Var} \cup \{*\}$.

Definition 16 Definition of $\mathrm{rk}_\sigma(e)$
1. If $\sigma \notin \mathrm{FV}(e) \cup \{*\}$ *then* $\mathrm{rk}_\sigma(e) := 0$.

2. For $\sigma \in \mathrm{FV}(e) \cup \{*\}$ *we define:*

$$\mathrm{rk}_\sigma(e) := \begin{cases} o(e) & \text{if } e \in \mathrm{Var} \cup \{0\} \\ \max\{\mathrm{rk}_\sigma(e_1), ..., \mathrm{rk}_\sigma(e_n)\} & \text{if } e = pe_1...e_n \\ \max\{o(\xi), \mathrm{rk}_\sigma(F), \mathrm{rk}_\xi(F) + 1\} & \text{if } e = \epsilon\xi F \\ \mathrm{rk}_\sigma(G) & \text{if } e = \lambda zG \end{cases} .$$

Note. This definition extends one in [15]. The only difference is with the clause $\mathrm{rk}_\sigma(pe_1...e_n) = \max\{o(p), \mathrm{rk}_\sigma(e_1), ..., \mathrm{rk}_\sigma(e_n)\}$, but $o(p)$ was redundant in [15] as shown by the Lemma 4.2(c) there.

$\mathrm{rk}_\sigma(e)$ is a measure of nesting of bound variables in subterms of e containing free variable σ , and rk_* takes account of all ϵ-subterms. More precisely,

Lemma 10

$$\mathrm{rk}_*(e) = \sup\{o(e), \mathrm{rk}_\sigma(F)+1 \ : \epsilon\sigma F \textit{ occurs in } e\} \tag{2}$$

Proof as in [15]. □

Next Lemmas establish properties of rank.

Lemma 11
a) $\mathrm{rk}_\sigma(e) < o(e) + \omega$; *in particular, if* $u \in \mathbb{B}_\alpha$ *then* $rks(u) < \omega \cdot \alpha$.
b) $X^\alpha \in \mathrm{FV}(e) \implies \omega \cdot \alpha \le \mathrm{rk}_X(e)$.
c) $o(P) \le \mathrm{rk}_*(P)$, *for each 1-term* P.

Proof. Easy induction on e, P.

Lemma 12
If $d(e) = 0$ *and* e *is not a* λ*-term then* $\mathrm{rk}_*(e) \le \sup\{\mathrm{rk}_\sigma(e) : \sigma \in \mathrm{FV}(e)\}$.

Proof: as in [15]. □

Lemma 13
If $\epsilon\xi F$ *is canonical then* $\mathrm{rk}_*(F) \le \mathrm{rk}_\xi(F)$ *and* $\mathrm{rk}_*(\epsilon\xi F) = \max\{o(\xi), \mathrm{rk}_\xi(F)+1\}$.

Proof: as in [15]. □

The next two lemmas show that the rank does not increase during computation.

Lemma 14
$\mathrm{rk}_\sigma(e[y/t]) \le \max\{\mathrm{rk}_\sigma(e), \mathrm{rk}_\sigma(t)\}$.

Proof.

Let $e = \epsilon\xi F$ with $y \in FV(e)$: remaining cases are treated easily: cf. [15] Lemma 4.5. By I.H.,

$\mathrm{rk}_\sigma(F[t]) \le \max\{\mathrm{rk}_\sigma(F), \mathrm{rk}_\sigma(t)\}$ and $\mathrm{rk}_\xi(F[t]) \le \max\{\mathrm{rk}_\xi(F), \mathrm{rk}_\xi(t)\} = \mathrm{rk}_\xi(F)$. The last equation holds by clause 1 in the definition of rank, since $\xi \notin \mathrm{FV}(t) \cup \{*\}$. Hence

$\mathrm{rk}_\sigma(e[t]) = \max\{o(\xi), \mathrm{rk}_\sigma(F[t]), \mathrm{rk}_\xi(F[t]) + 1\}$

$\le \max\{o(\xi), \mathrm{rk}_\sigma(F), \mathrm{rk}_\sigma(t), \mathrm{rk}_\xi(F) + 1\} = \max\{\mathrm{rk}_\sigma(e), \mathrm{rk}_\sigma(t)\}$

□

Lemma 15
If $e \hookrightarrow^1_S e'$ *then* $\mathrm{rk}_\sigma(e') \le \mathrm{rk}_\sigma(e)$.

Proof by induction on the definition of $\hookrightarrow^1_S$*:*
Let $\sigma \in \mathrm{FV}(e') \cup \{*\}$. (Otherwise $\mathrm{rk}_\sigma(e') = 0$.)
1.1. If e is a canonical 0-ϵ-term then $\mathrm{rk}_\sigma(e') = 0$, since $e' \in \mathbb{N}$.
1.2. If e is a canonical 1-ϵ-term $\epsilon X^\alpha F$ then one can assume $\sigma = *$ and obtain $\mathrm{rk}_\sigma(e) \ge \omega{\cdot}\alpha$ by the Lemma 13. Moreover, $lev(e') < \alpha$ and since e' is canonical, $o(e') < \omega{\cdot}\alpha$ by the Lemma 1. By the Lemma 11, $\mathrm{rk}_\sigma(e') < \omega \cdot \alpha \le \mathrm{rk}_\sigma(e)$.
2. $e = pe_1...e_n$ or $e = \lambda y F$:
2.1. $e = \mathsf{App}Pt$ and $e' = G[z/t]$ with $\epsilon X^\alpha F = P \hookrightarrow^1_S \lambda z G$. Then $\mathrm{rk}_\sigma(G) = \mathrm{rk}_\sigma(\lambda z G) \le \mathrm{rk}_\sigma(P)$ by the case 1, and
$\mathrm{rk}_\sigma(e') \overset{14}{\le} \max\{\mathrm{rk}_\sigma(G), \mathrm{rk}_\sigma(t)\} \le \max\{\mathrm{rk}_\sigma(P), \mathrm{rk}_\sigma(t)\} = \mathrm{rk}_\sigma(e)$.
2.2. otherwise: immediate from I.H.
3. $e = \epsilon\xi F$ and $e' = \epsilon\xi F'$ with $F \hookrightarrow^1_S F'$:
$\mathrm{rk}_\sigma(e') = \max\{o(\xi), \mathrm{rk}_\sigma(F'), \mathrm{rk}_\xi(F'){+}1\} \overset{\mathrm{IH}}{\le} \max\{o(\xi), \mathrm{rk}_\sigma(F), \mathrm{rk}_\xi(F){+}1\} = \mathrm{rk}_\sigma(e)$. □

Substitution of a variable by an appropriate canonical value also does not increase the rank.

Lemma 16
$\mathrm{rk}_\sigma(e[\eta/u]) \le \mathrm{rk}_\sigma(e)$, *for each* $u \in \mathbb{B}_{lev(\eta)}$.

Proof. $\mathrm{rk}_\sigma(e[u]) \overset{14}{\le} \max(\mathrm{rk}_\sigma(e), \mathrm{rk}_\sigma(u)\} = \mathrm{rk}_\sigma(e)$ since $\mathrm{rk}_\sigma(e) \ge \mathrm{rk}_\sigma(u)$. Indeed, $\mathrm{rk}_\sigma(u) = 0$ if $\sigma \ne *$ since u is closed. For $\sigma = *$ (and $\eta \in FV(e)$).
$\mathrm{rk}_*(e) \overset{2}{\ge} \omega \cdot lev(\eta) \overset{11(\mathrm{a})}{\ge} \mathrm{rk}_*(u)$. □

The next statement shows that our definition of rank is suitable: the rank decreases when the 'body' of a canonical ϵ-term is substituted by a canonical value.

Lemma 17
If $\epsilon\xi F$ *is canonical then* $\mathrm{rk}_*(F[u]) < \mathrm{rk}_*(\epsilon\xi F)$, *for each* $u \in \mathbb{B}_{lev(\xi)}$.

Proof: $\mathrm{rk}_*(F[u]) \overset{16}{\le} \mathrm{rk}_*(F) \overset{13}{\le} \mathrm{rk}_\xi(F) < \mathrm{rk}_*(\epsilon\xi F)$. □

Definition 17 $\mathrm{rk}(e) := \mathrm{rk}_*(e)$ *is called the* rank *of* e.

Definition 18 (Truncation to a given rank)
For each ϵ*-substitution* S *and* $r \in On$ *we set* $S_{\le r} := \{(e, u) \in S : \mathrm{rk}(e) \le r\}$.
Analogously we define $S_{\ge r}$, $S_{<r}$, $S_{>r}$.

Lemma 18
If S, S' *are* ϵ*-substitutions with* $S_{\le r} = S'_{\le r}$ *then* $|e|_S = |e|_{S'}$ *holds for all expressions* e *of rank* $\le r$.

Proof:
Since all subterms of an expression e have ranks $\le \mathrm{rk}(e)$, we have:
$\mathrm{rk}(e) \le r \;\Rightarrow\; \forall e' (\, e \hookrightarrow_S e' \;\Leftrightarrow\; e \hookrightarrow_{S'} e' \,)$.
Together with Lemma 15 this yields the assertion by induction on $d(e)$. □

5 The H-process

This section is similar to the Section 5 of [15].

We assume that $Cr_0, ..., Cr_N$ (with $N \in \mathbb{N}$) **is a fixed sequence of closed critical formulas**.

Definition 19
$F[\![x/n]\!] := F[x/n] \wedge \neg F[x/n{-}1] \wedge \ldots \wedge \neg F[x/0]$,
$F[\![X/T]\!] := F[X/T]$.

Definition 20
Let S be an ϵ-substitution:
$e \hookrightarrow_S \mathsf{TRUE}\,(\mathsf{FALSE}) \;:\Longleftrightarrow\; |e|_S \in \mathsf{TRUE}\,(\mathsf{FALSE})$.
$\mathcal{F}(S) := \{F[\![\xi/u]\!] : (\epsilon\xi F, u) \in S \,\&\, u \neq ?\,\}$
S is correct *iff $A \hookrightarrow_S$* TRUE *for all $A \in \mathcal{F}(S)$.*
S is solving *iff $Cr_I \hookrightarrow_S$* TRUE *for $I = 0, ..., N$. Otherwise S is* nonsolving.
$\overline{S} := S \cup \{(e, ?) : e$ *canonical* ϵ*-term* $\notin \mathrm{dom}(S)\}$ *is called the* standard extension of S, *cf. Section 3.*

Since ϵ-substitutions are defined only for canonical terms, all operations below are preceded by transforming arbitrary ϵ-term $\epsilon\xi F$ into its *canonical form* $\epsilon\xi|F|_S$.

Definition 21
Let S be an ϵ-substitution such that $\overline{S}$ is nonsolving. (Then $|Cr_I|_{\overline{S}} \in \mathsf{FALSE}$ for some $I \leq N$.)
Set $r_I := \mathrm{rk}(\epsilon\xi|F|_{\overline{S}})$, where $Cr_I = F_0 \to F[\epsilon\xi F]$.
$\mathrm{Cr}(S) := Cr_I$, *where $I \leq N$ is such that*
$|Cr_I|_{\overline{S}} \in \mathsf{FALSE} \,\&\, \forall J \leq N[\,|Cr_J|_{\overline{S}} \in \mathsf{FALSE} \Rightarrow r_I < r_J \vee (r_I = r_J \wedge I \leq J)]$.
Let $\mathrm{Cr}(S) = F_0 \to F[\epsilon\xi F]$*:*
$\epsilon\xi|F|_{\overline{S}}$ *is called the* H-term of S.
The H-value v *of S is defined as follows*
a) if $\iota(\xi) = 1$ and $F_0 = F[T]$ then $v := |T|_{\overline{S}}$,
b) if $\iota(\xi) = 0$, $F_0 = (s \neq 0)$, and $F = (s = \mathsf{S}x)$ then $v := |s|_{\overline{S}} - 1$,
c) if $\iota(\xi) = 0$ and $F_0 = F[t]$ then $v :=$ the unique $n \in \mathbb{N}$ with $|F|_{\overline{S}}[\![n]\!] \hookrightarrow_{\overline{S}}$ TRUE.

Definition 22 *If $\overline{S}$ is nonsolving then*
$\mathrm{H}(S) := (S \setminus \{(e, ?)\})_{\leq \mathrm{rk}(e)} \cup \{(e, v)\}$, *where e is the H-term and v the H-value of S.*

Lemma 19
Let S be an ϵ-substitution such that $\overline{S}$ is correct and nonsolving,
and let e be the H-term, v the H-value of S. Then the following holds:
(a) $(e, ?) \in \overline{S}$,
(b) $|e|_{\mathrm{H}(S)} = v \neq 0^{\iota(e)}$,
(c) $\overline{\mathrm{H}(S)}$ is correct.

Proof: Similar to the Lemma 5.1 in [15].

Definition 23
The H-process *(for $Cr_0, ..., Cr_N$) is defined as follows:*

$$S_0 := \emptyset, \quad S_{n+1} := \begin{cases} \mathrm{H}(S_n) & \textit{if } \overline{S_n} \textit{ is nonsolving} \\ \emptyset & \textit{otherwise} \end{cases}.$$

The H-process terminates *iff there exists an $n \in \mathbb{N}$ such that $\overline{S_n}$ is solving.*

Lemma 20
Let S be an ϵ-substitution such that $\overline{S}$ is correct, nonsolving, and $\mathrm{Cr}(S) = F[u] \to F[\epsilon\xi F]$. Then

a) $|u|_S \in \mathbb{B}_{lev(\xi)}$ and $F[|u|_S] \hookrightarrow_S \mathsf{TRUE}$.

b) The H*-value of S exists.*

Proof:

a) Since $\mathrm{Cr}(S) \hookrightarrow_{\overline{S}} \mathsf{FALSE}$, we have $F[u] \hookrightarrow_S \mathsf{TRUE}$. Now the assertion follows by the Lemmas 9c and Church-Rosser property.

b) If $lev(\xi) > 0$, apply (a). Let $lev(\xi) = 0$. Set $m := |t|_S$. Since $F[m] \hookrightarrow_S \mathsf{TRUE}$, $n := \min\{k : F[k] \hookrightarrow_S \mathsf{TRUE}\}$ exists and $n \leq m$. Hence $|F|_S[\![n]\!] \hookrightarrow_S \mathsf{TRUE}$. $\Box$

6 The proof of termination

This section is similar to the last section of [15]. We slightly simplify the termination proof. In fact it is possible to prove strong termination along the lines of [13], but this is not done here.

In this section S, $S_n, ...$ always denote ϵ-substitutions with $\{e \in \mathrm{dom}(S) : S(e) = ?\} = \emptyset$. For each pair (e, u) we set $\mathrm{rk}((e, u)) := \mathrm{rk}(e)$.

Definition 24 *Let S be an ϵ-substitution such that $\overline{S}$ is correct and nonsolving.*

Let e be the H-term and v the H-value of S.

We set $\mathrm{rk}(S) := \mathrm{rk}(e)$ *and* $\pi(S) := (e, v)$.

Note that if $\overline{S}$ is correct and nonsolving, then according to our general assumption on S we have $e \notin \mathrm{dom}(S)$ and $\mathrm{H}(S) = S_{\leq \mathrm{rk}(e)} \cup \{(e, v)\}$.

Definition 25 *An ϵ-substitution S is called* r-substitution ($r \in On$) *iff $\overline{S}$ is correct and* $\mathrm{rk}(\pi) < r$ *for all pairs $\pi \in S$, i.e. $S = S_{<r}$.*

Definition 26 *Let $r \in On$. An* r-process *is a sequence $(S_i)_{i<\nu}$ such that:*

- *$0 < \nu \leq \omega$,*
- *S_0 is an r-substitution,*
- *if $i < \nu$ and $\overline{S_i}$ is nonsolving and* $\mathrm{rk}(S_i) \geq r$ *then $i + 1 < \nu$ and $S_{i+1} = \mathrm{H}(S_i)$,*
- *if $i < \nu$ and ($\overline{S_i}$ is solving or* $\mathrm{rk}(S_i) < r$*) then $\nu = i + 1$.*

(In this case S_i is the last substitution, and $S_\nu = S_{i+1}$ is not defined.)

Remark. For each r-substitution S there is a unique r-process $(S_i)_{i<\nu}$ with $S = S_0$. We say that this process *starts with* S.

Lemma 21
If $(S_i)_{i\in\nu}$ is an r-process and $i \le j < \nu$ then $(S_i)_{\le r} \subseteq S_j$.

Proof: If $i{+}1 < \nu$ then $S_{i+1} = (S_i)_{\le \mathrm{rk}(e)} \cup \{(e,v)\}$ with $\mathrm{rk}(e) \ge r$, and therefore $(S_i)_{\le r} \subseteq S_{i+1}$. From this the claim follows by induction on j. $\square$

For the following set:

$$|e|_n := |e|_{\overline{S_n}}$$

Lemma 22 *Let γ be a limit ordinal and $(S_q)_{q<\gamma}$ be an increasing sequence of r_q-substitutions such that $\overline{S}_q$ is correct, $r_q < \alpha$ and $(S_{q'})_{<r_q} = (S_q)_{<r_q}$ for all $q < q' < \gamma$.*

Then $S^\gamma := \bigcup_{q<\gamma} S_q$ is an α-substitution.

Proof. Since the sequence is increasing, S^γ is a function. To prove that $\overline{S}^\gamma$ is correct, let $(\epsilon\xi A, u) \in S_q$. Then for each $v \in \mathbb{B}_{lev(\xi)}$ one has $rk(A[v]) < rk(\epsilon\xi A) \le r_q$ and therefore $|A[v]|_{q'} = |A[v]|_q = |A[v]|_{\overline{S}^\gamma}$ for $q < q' < \gamma$. Hence $|A[\![u]\!]|_q = |A[\![u]\!]|_{\overline{S}^\gamma}$. Since $\overline{S}_q$ is correct, we have $|A[\![u]\!]|_q = \mathsf{TRUE}$. $\square$

We use below the following notation and conventions:
$(S_i)_{i<\nu}$ is an r-process
$S{+} := \bigcup_{i<\nu}(S_i)_{\le r}$.
$(S{+}_i)_{i<\mu}$ is an $(r+1)$-process starting with $S{+}$.
$|e|^+_n := |e|_{\overline{S{+}_n}}$
$\Phi := S{+} \setminus S_k$

Lemma 23 *If S_k $(k < \nu)$ is an $r+1$-substitution, and no component of Φ is used in the process $(S{+}_i)_{i<\mu}$, then $\nu = k + \mu$ and for all l one has*

$$S{+}_l = S_{k+l} \bigcup \Phi \tag{3}$$

Proof by induction on l. Note that $S{+}$ is an $(r{+}1)$-substitution by Lemmata 21 ,22. Equation (3) holds for $l = 0$. Let $l < \mu$ and (3) hold.

By the definition of Φ, all pairs which are used in the computations of $|Cr_I|^+_l$, $|Cr_I|_{k+l}$ $(I \le N)$ and $\pi(S_l), \pi(S_{k+l})$ are contained in S_{k+l}. Hence we have $|Cr_I|^+_l = |Cr_I|_{k+l}$ $(I \le N)$ and $rk(S{+}_l) = rk(S_{k+l})$. So $\overline{S{+}_l}$ is solving, iff $\overline{S_{k+l}}$ is solving and $\mu = k + l + 1 = k + \nu$ in this case. Assume that S_{k+l} is not terminal. Then $\mathrm{rk}(S_{k+l}) \ge r$ and $\overline{S_{k+l}}$ is not solving. Let $Cr(S_{k+l}) = Cr_J,\ J \le N$.

Assume $\mathrm{rk}(S_{k+l}) = r$. Then $\pi(S_{k+l}) \in S_{k+l+1}$, while J cannot be used in the process $(S{+}_i)_{i<l+1}$, and the value $\pi(S_{k+l})$ was used to terminate this process. Hence by the definition of k, $\pi(S_{k+l}) \in S_k \subseteq S_{k+l}$ which contradicts Lemma 19 (a).

In the remaining case $\mathrm{rk}(S_{k+l}) \ge r{+}1$ one has

$\pi(S{+}_l) = \pi(S_{k+l}) \notin \Phi$, $\mathrm{rk}(S{+}_l) = \mathrm{rk}(S_{k+l})$ and $\mathrm{H}(S{+}_l) = S{+}_{l+1} = S_{k+l+1} \cup \Phi$ which concludes the proof. $\square$

Theorem 3
Let $(S_i)_{i\in\omega}$ be an r-process. Then (by adding pairs of rank r) S_0 can be extended to an $(r{+}1)$-substitution $S{+}$ such that the $(r{+}1)$-process beginning with $S{+}$ is infinite.

Proof. Using the notation of the Lemma 23 assume that the process $(S+_i)_{i<\nu}$ is finite. Then there exists a k satisfying conditions of the Lemma 23 and hence the original process is finite. *Contradicition.* □

Theorem 4
The 0-process Π *beginning with the empty substitution* $\emptyset$ *terminates in a solving substitution.*

Proof: Obviously it suffices to prove that Π is finite (terminates).
For contradiction we assume that Π is infinite.
Below we define substitutions S^r for all $r \in On$ such that:
(1) $S^0 = \emptyset$,
(2) S^r is an r-substitution,
(3) the r-process starting with S^r is infinite,
(4) $S^q \subseteq S^r$, for all $q < r$.

Let $R := \max\{\text{rk}(Cr_I) : I \leq N\} + 1$. Then $\text{rk}(S) < R$ for each substitution S. But on the other hand $\text{rk}(S^R) \geq R$, since by (3) the R-process starting with S^R is infinite. *Contradiction.*

Definition of S^r by transfinite recursion on r:
$S^0 := \emptyset$,
$S^{r+1} := (S^r)+$ (cf. Theorem 3),
$S^r := \bigcup_{q<r} S^q$, if $r \in Lim$.

In parallel with that definition we prove by transfinite induction on r that S^r satisfies the above conditions (2),(3),(4). The successor step is settled by theorem 3 and the limit case by the Lemma 22. □

7 $\mathbf{\Delta^1_1}$-analysis and its ϵ-formulation

7.1 System DA for $\mathbf{\Delta^1_1}$-analysis

By $\mathbf{\Delta^1_1}$-*analysis* we mean a subsystem of the second order arithmetic with full induction axiom and Δ^1_1-comprehension rule.

To fix notation, we consider the following Hilbert-style formulation **DA** of $\mathbf{\Delta^1_1}$-analysis.

The language **L2** of the system is the standard second order language obtained from the language LA (Section 2) by dropping levels of 1-variables.

Axioms of the system **DA** are obtained from the axioms of RA different from the second order existential axioms by dropping levels of 1-variables. Inference rules of **DA** are modus ponens, $\exists \rightarrow$ (cf. Section 2) and the following Δ^1_1-Comprehension Rule:

$$(\Delta) \quad \frac{\forall x(\exists Y A[x,Y] \leftrightarrow \forall Z B[x,Z])}{F[\lambda x(\exists Y A[x,Y])] \rightarrow \exists X F[X]}$$

where formulas $A[a,D]$ and $B[a,D]$ do not contain bound predicate variables. Note that the arithmetic comprehension axiom can be derived from the Δ^1_1-Comprehension Rule: take redundant variables Y, Z and $B = A$.

The variable η in the rule $\exists\rightarrow$ (cf. Section 2) is an *eigenvariable*, and the formula $\exists\xi F[\xi]$ is the *main formula.*

As shown in [19] (Theorem 23.5), this system has a predicative interpretation and its proof-theoretic ordinal is $\phi\omega 0$, where ϕ is the fixpoint ordinal function.

Definition 27 *The* height $h(d)$ *of a derivation* d *in* **DA** *is the maximal number of rules* Δ *in one branch.*

More precisely, $h(d) = 0$ *if* d *is an axiom,* $h(d) = h(d')$ *if* d *is obtained from* d' *by a rule other than* Δ*, and* $h(d) = h(d') + 1$ *if* d *is obtained from* d' *by the rule* Δ*.*

The endpiece *of a* **DA***-derivation is its part from the endformula up to and including the conclusions of lowermost rules* Δ *.*

Definition 28 d *is a* pure variable *derivation ([16]), if the eigenvariable of any rule* $\exists\rightarrow$ *occurs only above its conclusion. Pure variable derivation is* non-redundant *if each free variable is an eigenvariable or occurs in the endsequent.*

Each derivation can be made non-redundant by renaiming eigenvariables and substituting suitable constants for redundant variables. We consider below only non-redundant derivations in **DA**.

Note. Eigenvariables of different rules in a non-redundant derivation are distinct and different from the free variables of the endsequent.

7.2 System RA($\boldsymbol{\Sigma}$)

In this section we introduce a system **RA**($\boldsymbol{\Sigma}$), where $\Sigma =< \varsigma_0, \ldots, \varsigma_n >$ is a finite list of new objects called *ordinal variables.* We translate a derivation in **DA** into a derivation in **RA**($\boldsymbol{\Sigma}$) with a suitable Σ. Predicate variables of this system will have *levels* from Σ. These ordinal variables are assigned values during an H-process (cf. Sections 5,8.1).

7.2.1 Formulation of the system

Let us fix a list $\Sigma =< \varsigma_0, \ldots, \varsigma_n >$ of ordinal variables. The language **LR**($\boldsymbol{\Sigma}$) is defined exactly as LA in section 2 with the only difference: predicate variables have levels from Σ (which are used as their superscripts): $lev(X^\varsigma) == \varsigma$. The *ordering* $<$ on the set $\Sigma \cup \{0\}$ is defined as follows:

$0 < \varsigma_i$ for each i; $\quad \varsigma_i < \varsigma_j$ iff $i < j$.

The levels of expressions are defined exactly as in the section 2. We use the same syntactic conventions and definitions as in that section.

To define derivations in **RA**($\boldsymbol{\Sigma}$), we define first an auxiliary system $\mathbf{RA^{aux}}(\boldsymbol{\Sigma})$ having the same axioms and inference rules as **DA**, except the postulates for the second order quantifier. $\exists^2$-rule still has the form

$$\frac{F[Y^\varsigma] \rightarrow G}{\exists X^\varsigma F[X^\varsigma] \rightarrow G}\ \exists^2$$

with the standard proviso: the *eigenvariable* Y^ς does not occur in the conclusion.

$$\begin{array}{c}
\backslash|/ \\
\dfrac{F_n[\eta, \eta_{n-1}] \to C_n}{\exists \xi_n F_n[\xi_n, \eta_{n-1}] \to C_n} \\
\backslash|/ \\
\dfrac{F_i[\eta_i, \eta_{i-1}] \to C_i}{\exists \xi_i F_i[\xi_i, \eta_{i-1}] \to C_i} \\
\backslash|/ \\
\dfrac{F_1[\eta_0, \eta_1] \to C_1}{\exists \xi_1 F_1[\xi_1, \eta_0] \to C_1} \\
\backslash|/ \\
\dfrac{F_0[\eta_0] \to C_0}{\exists \xi_0 F_0[\xi_0] \to C_0}
\end{array}$$

Figure 1:

Instead of Δ^1_1-Comprehension Rule there is the following

$$\exists^2 - axiom : \qquad\qquad F[T] \to \exists X^\varsigma F[X^\varsigma]$$

where T is a lambda-term of level $< \varsigma$ or a free variable (of any level, not necessarily $\leq \varsigma$).

Definition 29 *Let* d *be a non-redundant derivation in* $\mathbf{RA^{aux}(\Sigma)}$ *(cf. end of the section 7). We say that an occurrence of a free (individual or predicate) variable* η *is* accessible *from a* $\exists \to$ *rule* Φ *in* d *iff there is a sequence* $\Phi_0, \ldots, \Phi_n$ *(*$n \geq 0$*) of* $\exists$*-rules in* d *with main formulas* $\exists\xi_0 F_0, \ldots, \exists\xi_n F_n$ *and eigenvariables* $\eta_0, \ldots \eta_n$ *(*accessibility analysis*), such that (cf. Figure 7.2.1):*

Φ_0 *is* Φ*;*

formula $\exists\xi_i F_i$ *contains free the eigenvariable* η_{i-1} *of the rule* Φ_{i-1} *for* $i = 0, \ldots, n-1$*, and*

$\eta = \eta_n$ *is the eigenvariable of the rule* Φ_n*.*

Note (to be used in the Section 7.3.3). If the main formulas $\exists\xi_i F_i[\xi_i]$ of the rules Φ_i are translated as $F_i[\epsilon\xi_i F_i]$, and eigenvariables η_i are replaced by $\epsilon\xi_i F_i$, then η is substituted by $\epsilon\xi_n F_n$ which contains all $\epsilon\xi_i F_i$, including $\epsilon\xi_0 F_0$ as subterms.

Definition 30 *The derivation in* $\mathbf{RA^{aux}(\Sigma)}$ *is* a derivation in $\mathbf{RA(\Sigma)}$ *iff its* $\exists^2$*-axioms*

$$F[T] \to \exists X^\varsigma F[X^\varsigma]$$

satisfy the following majorization proviso*:*

If the term T *contains a free predicate variable* Y^ϑ *with* $\vartheta > \varsigma$ *, and* Y^ϑ *is accessible from a rule* Φ *then all bound predicate variables in the main formula of* Φ *have levels* $\geq \vartheta$*.*

This concludes description of the system $\mathbf{RA(\Sigma)}$. The majorization proviso is connected with embedding of the system **DA** into ramified analysis (cf. [19] and the section 7.2.2 below).

The following Extensionality Lemma is an analogue of Theorem 22.11 from [19].

Lemma 24 For any formulas F, G, H of $\mathbf{LR}(\mathbf{\Sigma})$ the formula

$$\forall x(G \leftrightarrow H) \ \rightarrow \ F[\xi/\lambda x G] \leftrightarrow F[\xi/\lambda x H]$$

is derivable in $\mathbf{RA}(\mathbf{\Sigma})$.

The **proof** is by induction on F. Note that $\exists^2$-axioms used in this derivation are only of the kind

$$G[B^{\vartheta}] \rightarrow \exists Y^{\vartheta} G[Y^{\vartheta}]$$

so that majorization proviso is vacuously satisfied. □

7.2.2 Embedding of DA into RA(Σ)

In this section we define a translation of derivations in **DA** into $\mathbf{RA}(\mathbf{\Sigma})$. It mimicks the proof of Theorem 22.14 from [19]. Recall that $\Sigma =< \varsigma_0, \ldots, \varsigma_n >$.

Definition 31 *An* m-stratification *($m \leq n$) of an expression e of* **L2** *is any result of stratifying bound predicate variables in it by assigning level ς_m, and stratifying free predicate variables by assigning levels $\geq \varsigma_m$. All occurrences of the same variable in a formula are assigned the same level.*

Note. m-stratifications of axioms and inference rules of **DA** other than the rule Δ are axioms and rules of $\mathbf{RA}(\mathbf{\Sigma})$.

For every non-redundant derivation d of a formula G in **DA**, for every m such that $m \leq n$, $m \leq h(\mathsf{d})$ (section 1.1) and every m-stratification G_m of G, we define a derivation d_m of G_m in $\mathbf{RA}(\mathbf{\Sigma})$ such that bound variables in d_m have levels $\leq \varsigma_m$. The definition is given by induction on $h(\mathsf{d})$.

Induction base. $h(\mathsf{d}) = 0$. Stratify all bound predicate variables and all eigenvariables by ς_m. Stratify all variables free in G in the same way as they are stratified in G_m.There is no conflict with the stratification of eigenvariables: they are different from free variables of G since d is non-redundant. All variables are now stratified. Since d does not contain the rule Δ , the resulting figure d_m is a derivation of G_m by the previous Note.

Induction step. Let us stratify the endpiece explicitly, and then apply induction hypothesis to derive stratifications of the sequents situated immediately above the endpiece.

Stratify all bound predicate variables and all eigenvariables occurring in the endpiece by ς_m. Stratify all variables free in G in the same way as they are stratified in G_m. Again, there is no conflict since d is non-redundant. Note that d has a form:

$$\begin{array}{ll} & \dfrac{\mathsf{d}' : \forall x(\exists Y A[x,Y] \leftrightarrow \forall Z B[x,Z])}{F[\lambda x(\exists Y A[x,Y])] \rightarrow \exists X F[X]} \ \Delta \\ \textit{endpiece}: & \qquad\qquad \backslash | / \\ \mathsf{d}: & \qquad\qquad G \end{array}$$

where Δ is a rule Δ , and hence $h(\mathsf{d}') < h(\mathsf{d})$. We have to derive the stratification of the conclusion of a lowermost rule Δ, that is the formula

$$F_m[\lambda x(\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}])] \rightarrow \exists X^{\varsigma_m} F_m[X^{\varsigma_m}] \tag{1}$$

$$\dfrac{\dfrac{\dfrac{\overset{\exists^2 axiom}{B_m[x, Z^{\varsigma_{m-1}}] \to \exists Z^{\varsigma_m} B_m[x, Z^{\varsigma_m}]}}{\exists Z^{\varsigma_{m-1}} B_m[x, Z^{\varsigma_{m-1}}] \to \exists Z^{\varsigma_m} B_m[x, Z^{\varsigma_m}]}}{\forall Z^{\varsigma_m} B_m[x, Z^{\varsigma_m}] \to \forall Z^{\varsigma_{m-1}} B_m[x, Z^{\varsigma_{m-1}}]} \quad \overset{\mathsf{d}''_{m-1}}{\forall Z^{\varsigma_{m-1}} B_m[x, Z^{\varsigma_{m-1}}] \to \exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}]}}{\forall Z^{\varsigma_m} B_m[x, Z^{\varsigma_m}] \to \exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}]}$$

$$\dfrac{\mathsf{d}'_m : \ \forall x(\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}] \leftrightarrow \forall Z^{\varsigma_m} B_m[x, Z^{\varsigma_m}]) \quad \forall Z^{\varsigma_m} B_m[x, Z^{\varsigma_m}] \to \exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}]}{\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}] \to \exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}]} \begin{array}{l} \forall elim., \\ \quad cut \end{array}$$

$$\forall - introduction$$

$$\dfrac{\dfrac{\forall x(\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}] \leftrightarrow \exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}])}{F_m[\lambda x(\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}])] \to F_m[\lambda x(\exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}])]} \quad \overset{\exists^2 axiom}{\to \exists X^{\varsigma_m} F_m[X^{\varsigma_m}]}}{F_m[\lambda x(\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}])] \to \exists X^{\varsigma_m} F_m[X^{\varsigma_m}]}$$

Figure 2:

from suitable stratifications of the premise of that rule. The derivation is shown schematically in the Figure 7.2.2 and explained below.

Note that free variables of the premise of the rule (Δ) occur in the conslusion (since d is non-redundant), and hence they received levels $\geq \varsigma_m$. By the induction hypothesis for d' there is a derivation d'_m of a formula

$$\forall x(\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}] \leftrightarrow \forall Z^{\varsigma_m} B_m[x, Z^{\varsigma_m}]) \tag{2}$$

such that bound variables in d'_m have levels $\leq \varsigma_m$. Now we apply the induction hypothesis once more to d' and $m-1$ instead of m to obtain a derivation d''_{m-1} of the formula

$$\forall x(\exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}] \leftrightarrow \forall Z^{\varsigma_{m-1}} B_m[x, Z^{\varsigma_{m-1}}]) \tag{3}$$

such that bound variables in d''_{m-1} have levels $\leq \varsigma_{m-1}$. We use the same notation A_m, B_m for the stratifications of $A[x, Y], B[x, Z]$ in (1,2,3) since these formulas contain no bound predicate variables except Y, Z respectively, and their free variables are stratified in exactly the same way. Formula (1) is derived from an $\exists^2$-axiom

$$F_m[\lambda x(\exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}])] \to \exists X^{\varsigma_m} F_m[X^{\varsigma_m}] \tag{4}$$

and a formula

$$F_m[\lambda x(\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}])] \to F_m[\lambda x(\exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}])] \tag{5}$$

The majorization proviso in the $\exists^2$-axiom 4 is satisfied since all the eigenvariables and bound variables of the endpiece have the same level ς_m, and the proviso requires only that the levels of some bound variables majorize levels of some eigenvariables. The formula (5) is derived by the Extensionality Lemma from the formula

$$\forall x(\exists Y^{\varsigma_m} A_m[x, Y^{\varsigma_m}] \leftrightarrow \exists Y^{\varsigma_{m-1}} A_m[x, Y^{\varsigma_{m-1}}]) \tag{6}$$

The latter is derived in the following way. Direction $\leftarrow$ is obtained by the rule $\exists^2$ from an $\exists^2$-axiom with vacuously satisfied majorization condition.

Note, however, that direction $\rightarrow$ may not be derivable in a similar way, since the majorization proviso can be violated in

$$B_m[x, Z^{\varsigma_m}] \rightarrow \exists Z^{\varsigma_{m-1}} B_m[x, Z^{\varsigma_{m-1}}]) :$$

eigenvariable Z^{ς_m} to be used as a side term may be accessible from a quantifier rule for the exterior $\forall$, the main formula of which may contain ς_{m-1}. The direction $\rightarrow$ in (6) is derived from formulas (2) and (3) and a formula

$$\forall x(\forall Z^{\varsigma_m} B_m[x, Z^{\varsigma_m}] \rightarrow \forall Z^{\varsigma_{m-1}} B_m[x, Z^{\varsigma_{m-1}}]) \tag{7}$$

which is derived from an $\exists^2$-axiom in the same way as the direction $\rightarrow$ in (6). To verify that the whole Figure 7.2.2 is a derivation , it remains only to check the majorization provisos for $\exists^2$-axioms of $\mathsf{d'}_m$ and $\mathsf{d''}_{m-1}$. Consider such axiom

$$G[T] \rightarrow \exists X^{\varsigma_p} G[X^{\varsigma_p}], \tag{8}$$

an eigenvariable A^{ς_q} with $q > p$ occurring in T and a rule Φ in our figure such that A^{ς_q} is accessible from Φ. If Φ is in the subderivation $\mathsf{d'}_m$ or $\mathsf{d''}_{m-1}$ considered at this moment, refer to the proviso for the same derivation. Note also that the bound variables and hence eigenvariables in the whole derivation have levels $\leq \varsigma_m$, and that the part of our figure below $\mathsf{d'}_m$, $\mathsf{d''}_{m-1}$, that is below (2,3) contains bound variables only of the level ς_m. Hence, we have $q \leq m$, and if Φ is below (2, 3), then all bound variables in its main formula have levels $\geq m \geq q$ as required. This concludes the derivation of the formula (1) and thus definition of the derivation d_m.

Let d be a derivation in **DA** and

$$p = h(\mathsf{d}).$$

Consider list $\Sigma =< \varsigma_0, \ldots, \varsigma_p >$ of ordinal variables. We define a translation $\mathsf{d*}$ of d into $\mathbf{RA}(\mathbf{\Sigma})$ by

$$\mathsf{d}* = \mathsf{d}_p.$$

7.3 The system $\mathbf{RA}\epsilon(\mathbf{\Sigma})$

$\mathbf{RA}\epsilon(\mathbf{\Sigma})$ is a reformulation of the system $\mathbf{RA}(\mathbf{\Sigma})$ in terms of Hilbert's ϵ-symbol obtained from $\mathbf{RA}(\mathbf{\Sigma})$ in the same way as the system $RA\epsilon$ is obtained from RA, i.e. by the standard translation

$$\exists x F[x] = F[\epsilon x F[x]]; \qquad \exists X F[X] = F[\epsilon X F[X]]$$

and substitution of ϵ-terms for eigenvariables in the same way as it is done in [7, 15].

7.3.1 The language $\mathbf{LR}\epsilon(\mathbf{\Sigma})$

Let $\Sigma =< \varsigma_0, \dots, \varsigma_p >$ be a list of ordinal variables. The language is obtained from the language $LR\epsilon$ by changing stratification: predicate variables of $\mathbf{LR}\epsilon(\mathbf{\Sigma})$ have levels in $\mathbf{\Sigma}$. All remaining definitions from the section 1.1 are preserved and Lemmata 1,2,3 are carried over together with their proofs.

Definition 32 *A subterm or subformula (occurrence) in an expression e is* exterior *if it is not in the scope of an ϵ-symbol.*

7.3.2 The system $\mathbf{RA}\epsilon(\mathbf{\Sigma})$

Derivable objects of $\mathbf{RA}\epsilon(\mathbf{\Sigma})$ are closed formulas in the language $\mathbf{LR}\epsilon(\mathbf{\Sigma})$.

Axioms of $\mathbf{RA}\epsilon(\mathbf{\Sigma})$ except second order critical formulas have exactly the same form as in the system $RA\epsilon$ (taking into account difference in the stratification).

Second order critical formulas have a form

$$F[T] \to F[\epsilon X^{\varsigma} F[X^{\varsigma}]]$$

where T is a lambda-term of level $< \varsigma$ or a 1-epsilon-term of a level $\leq \varsigma$, and T satisfies the following *majorization proviso*:

for each exterior closed ϵ-subterm $\epsilon Y^{\varsigma} H$ of T (including T itself), if $\vartheta > \varsigma$ then all predicate variables in $\epsilon Y^{\varsigma} H$ have levels $\geq \vartheta$.

The only rule of inference is modus ponens.

7.3.3 Embedding of $\mathbf{RA}(\mathbf{\Sigma})$ into $\mathbf{RA}\epsilon(\mathbf{\Sigma})$

Translation $F| \to F^*$ of the language $\mathbf{LA}(\mathbf{\Sigma})$ into $\mathbf{LA}\epsilon(\mathbf{\Sigma})$ is defined exactly as in the section 2.

Let d be arbitrary non-redundant derivation in $\mathbf{RA}(\mathbf{\Sigma})$. From now on we assume that **the last formula of d is closed**: otherwise one can replace every variable ξ of the endformula by $0^{\iota(\xi)}$.

We define a derivation d' of the formula $F*$ in $\mathbf{RA}\epsilon(\mathbf{\Sigma})$ in the following way.

(1) All formulas F are replaced by F^*.

(2) Translations of induction axioms are derived from first order critical formulas, minimality axioms and equality axioms like in [7].

(3) an eigenvariable of every $\exists$-rule with main formula $\exists \xi F[x]$, is replaced by $\epsilon \xi F[x]$.

In other words, every formula F is transformed into a formula

$$F' = F^*[\vec{\eta}/\vec{\eta^*}], \qquad \vec{\eta} = FV(F) \tag{9}$$

Under our assumptions, every variable $\eta \in \vec{\eta}$ is an eigenvariable of an $\exists$-rule situated below given occurrence of F, and hence

$$\eta^* = \epsilon \eta H[\vec{\xi}/\vec{\xi^*}], \qquad \vec{\xi} = FV[\epsilon \eta H] \tag{10}$$

where $\epsilon \eta H$ is the main formula of that $\exists$-rule. All the ϵ-terms η^*, ξ^* are closed.

Note that under this translation the first order exsistential axioms are transformed into first order critical formulas and quantifier rules are eliminated.

It remains to verify that the second order existential axioms are transformed into the second order critical formulas.

Lemma 25 *If F is a formula in* d, *and a variable Y occurs in F', then $Y \in BV(F)$ or there are a $\xi \in FV(F)$ and an $\exists$-rule Φ below F such that ξ is accessible from Φ and Y occurs bound in the principal formula of Φ.*

Proof by induction on the number n of $\exists$-rules below F. If n=0, then F contains no free variables, $F' = F^*$ and all variables of F^* occur in F. Otherwise by 9 the variable Y occurs in F or in $\vec{\eta}^*$. Then by 10 it occurs bound in the main formula of the rule with the eigenvariable η or, by induction hypothesis, in the main formula of a rule with an eigenvariable $\xi \in \vec{\xi}$. In both cases this rule Φ is as required. $\Box$

It is easy now to derive the majorization proviso for the second order critical formulas in d' from the majorization proviso for the second order existential axioms in d. Indeed, consider such an axiom

$$F[T] \to \exists X^{\varsigma} F[X^{\varsigma}]$$

in d. It is transformed into

$$F'[T'] \to F'[\epsilon X^{\varsigma} F']$$

Assume that T is a λ-term. Then $lev(T) < \varsigma$, and since the level is preserved by the *-translation and substitution of ϵ-terms for variables of the same level, one has $lev(T') = lev(T) < \varsigma$. By the previous Lemma 25 all predicate variables in T covered by the $\mathbf{RA}\epsilon(\mathbf{\Sigma})$-proviso occur also in main formulas of d covered by the $\mathbf{RA}(\mathbf{\Sigma})$-proviso. $\Box$.

8 Epsilon substitution method for $\mathbf{RA}\epsilon(\mathbf{\Sigma})$

In this section we define H-process for any finite system of critical formulas in $\mathbf{RA}\epsilon(\mathbf{\Sigma})$. Due to majorization proviso for critical formulas (cf. 7.3.2) it will be possible to interpret ordinal variables by ordinals less than ω^ω depending on a given ϵ-substitution S for $RA\epsilon$, and then use the definition of a computation and H-step for $RA\epsilon$ (Section 5). The present section uses approach developed in the Section 7 of [15].

We consider the system $\mathbf{RA}\epsilon(\mathbf{\Sigma})$ from the section 7.3, where $\Sigma =< \varsigma_0, \ldots, \varsigma_p >$ is a list of ordinal variables. From now on we fix a finite system $E = \{Cr_I \mid I = 1 \ldots N\}$ of critical formulas in $\mathbf{RA}\epsilon(\mathbf{\Sigma})$.

ς_p will be always interpreted by ω^p: $S(\varsigma_p) = \omega^p$.

Assume that $S(\varsigma_i)$ is defined for all $j > i$. To define interpretation $S(\varsigma_i)$ of ς_i, consider all exterior closed ϵ-subterms

$$U = \epsilon Y^{\varsigma_j} G, \; j > i$$

of side terms T of second order critical formulas having levels $\leq \varsigma_i$. Due to the the majorization proviso they contain ordinal variables only of levels $> \varsigma_i$, and so by induction hypothesis their values $|U|_{\overline{S}}$ can be computed. Then it turns out to be possible to make the interpretation $S(\varsigma_i)$ of ς_i greater than levels of all these $|U|_{\overline{S}}$. Such a choice of $S(\varsigma_i)$ will ensure that

S-values of side terms T of critical formulas of level ς_i will have levels $< S(\varsigma_i)$, so that the critical formulas of $\mathbf{RA}\epsilon(\mathbf{\Sigma})$ are translated into the critical formulas of $RA\epsilon$. This scheme uses the idea of assigning levels from the embedding of $\mathbf{\Delta^1_1}$-analysis into Ramified Analysis in [19], Theorem 22.14.

8.1 Assignment of Ordinals to Ordinal Variables

Let S be an ϵ-substitution. We define ordinals $\sigma_j = S(\varsigma_i)$ by induction on $p - i$.

$$\sigma_p = S(\varsigma_p) := \omega^p$$

Suppose that $\sigma_j = S(\varsigma_j)$ is defined for all $j > i$. For any close ϵ-term U of the language $\mathbf{RA}\epsilon(\mathbf{\Sigma})$containing only ordinal variables ς_j for $j > i$ consider the result

$$U^* = U[\varsigma_{i+1}/S(\varsigma_{i+1}), \ldots, \varsigma_p/S(\varsigma_p)]$$

of replacing $\varsigma_j,\ j > i$ by σ_j).This is an ϵ-term of the language $\mathbf{RA}\epsilon$, and by the Lemma 2 it has a value $|U^*|_{\overline{S}}$ under ϵ-substitution $\overline{S}$. Consider the set

$\mathcal{T}_i :=\ \{U :\ U = \epsilon Y^{\varsigma_j} G$ is closed, $j > i$ and U is an exterior subterm of a side term of some critical formula having level $\leq \varsigma_i\}$

By the majorization proviso, terms $U \in \mathcal{T}_i$ contain only ordinal variables ς_k with $k > i$. For every such term $U = \epsilon Y^{\varsigma_j} G$ one has

$$U^* = \epsilon Y^{S(\varsigma_j)} G' \qquad \mathit{and} \qquad lev(|U^*|_{\overline{S}}) < lev(S(\varsigma_j)) \tag{11}$$

since $|U^*|_{\overline{S}} \in \mathbb{B}_{S(\varsigma_j)}$. Note that $\mathcal{T}_p = \emptyset$.

Suppose now that the ordinals $\sigma_j,\ j > i$ satisfy the following conditions:

(a) $0 < \sigma_j < \sigma_{j+1} < \ldots < \sigma_p = \omega^p$;

(b) $\sigma_j = \omega^j \cdot \beta_j$ with $\beta_j > 0$;

(c) $\sigma_j > lev(|U^*|_{\overline{S}})$ for every $U \in \mathcal{T}_j$.

Set

$$\alpha_i := \max(lev(|U^*|_{\overline{S}}) : U \in \mathcal{T}_i)$$

and prove

$$\alpha_i < \sigma_{i+1}. \tag{12}$$

Let $U = \epsilon Y^{\varsigma_j} G \in \mathcal{T}_i$. If $j = i + 1$, then $lev(|U^*|_{\overline{S}}) < \sigma_{i+1}$ by (11). If $j > i + 1$ then $U \in \mathcal{T}_{i+1}$ and $lev(|U^*|_{\overline{S}}) < \sigma_{i+1}$ by the I.H. (c) which concludes the proof of 12.

Now set

$$\sigma_i = S(\varsigma_i) := \min \sigma(\alpha_i < \sigma < \sigma_{i+1}\ \wedge\ \exists \beta > 0(\sigma = \omega^i \cdot \beta)) \tag{13}$$

Such a σ of the form $\omega^i \cdot \beta$ between α_i and σ_{i+1} always exists: it is a property of ordinals of the kind $\omega^k \cdot \gamma$ (cf. [19], Theorem 14.12).

We now have (a),(b) for $j \geq i$ by (13), and (c) for $j = i$ by (12).

This concludes the definition of the *ordinal assignment* $S(\Sigma)$.

Note that (a),(b),(c) hold for all $i = o, \ldots, p$.

8.2 Termination of the H-process for $\mathbf{RA}\epsilon(\mathbf{\Sigma})$

If $e[\varsigma_0, \ldots, \varsigma_p]$ is an expression in the language $\mathbf{LR}\epsilon(\mathbf{\Sigma})$, and an ϵ-substitution S is fixed, then we write

$$e^* := e[\varsigma_0/\sigma_0, \ldots, \varsigma_p/\sigma_p] \qquad where \quad \sigma_i = S(\varsigma_i),$$

and transfer to $\mathbf{RA}\epsilon(\mathbf{\Sigma})$ the whole set-up of section 5 using the substitution *. In particular, $e \hookrightarrow_S e'$ iff $e^* \hookrightarrow_S e'$, $|e|_S =_{def} |e^*|_S$, $\mathrm{rk}(e) =_{def} \mathrm{rk}(e^*)$.

The H-term and H-value of S (for E) are by the definition, the H-term and H-value of S for E^*, and the next substitution H(S) is defined exactly as in the section 5. Lemmata 19,20 guarantee that $H(S)$ is defined and $\overline{H(S)}$ is correct.

This allows to define an H-*process* for $\mathbf{RA}\epsilon(\mathbf{\Sigma})$ exactly as it is defined for RA in the section 5. Let us verify that the termination proof in the section 6 goes through for $\mathbf{RA}\epsilon(\mathbf{\Sigma})$.

An r-substitution and an r-process (starting with a given r-substitution) are already defined there. Hence one can use Lemmata 21,22.

The proof of the Lemma 23 is almost unchanged: note that under the conditions of that Lemma equation 3 implies that $S +_l (\Sigma) = S_{k+l}(\Sigma)$.

Theorem 3 is obtained as before, and in the Theorem 4 one sets $R = \omega \cdot (\omega^p + 1)$. This concludes the proof of termination.

Theorem 5 *The 0-process for* $\mathbf{RA}\epsilon(\mathbf{\Sigma})$ *beginning with the empty substitution* $\emptyset$ *terminates in a solving substitution.*

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