Cut-elimination for Simple Type Theory with an Axiom of Choice

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Abstract

We present a cut-elimination proof for simple type theory with axiom of choice modeled after Takahashi's proof of cut-elimination for simple type theory with extensionality. The same proof works when types are restricted, for example for second-order classsical logic with axiom of choice.

1 Introduction: Simple Type Theory with ϵ-symbol

This is an extension of the author's previous result [2] to simple type theory. The proof is new: it is modeled after Takahashi's proof of cut-elimination for simple type theory with extensionality. We closely follow presentation of G. Takeuti in §20 − 21 of [6] and preserve all definitions and syntactic conventions. Some ideas from the work of Yasuhara [9] inspired by the author's review [3] of [8] are used.

The rest of the Introduction describes the formal system. A cut-elimination proof is given in the section 2. The end of this section indicates how to adapt this proof to obtain more elgant argument for the second order case than in [2] . There seem to be no obstacles to extension of the standard cut-elimination proof for a system without extensionality (cf. the end of §21 in [6]) to simple type theory with ϵ-symbol but without extensionality. However a value of such a result is not clear.

Types are 0 and $[\tau_1, \dots \tau_k]$, where $\tau_1, \dots \tau_k$ are types. There are individual constants, function constants (for functions from individuals to individuals) and predicate constants of all types $\tau \neq 0$, as well as free variables $a_0^\tau, a_1^\tau, \dots$ and bound variables $x_0^\tau, x_1^\tau, \dots$ of each type τ. Logical symbols are $\neg, \wedge, \forall$ and Hilbert's epsilon-symbol ϵ. In addition to predicate variables and abstracts one has ϵ-terms of all types:

individual constants are terms of type 0;

free variables of type τ and predicate constants of type τ are terms of type τ;

if f is i-ary function constant and $t_1, \dots, t_i$ are terms of type 0, then $f(t_1, \dots, t_i)$ is a term of type 0;

if $A(a_0^{\tau_0}, \dots, a_k^{\tau_k})$ is a formula then $\{x_0, \dots, x_k\}A(x_0, \dots, x_k)$ is a term (called an abstract) of the type $[\tau_0, \dots, \tau_k]$ provided a's are distinct free variables and x's are distinct new bound variables of suitable types.

if $A(a^\tau)$ is a formula then $\epsilon x A(x)$ is a term of type τ provided x is a new bound variable of type τ.

if α is a predicate constant or a free variable or an ϵ-term of type $[\tau_1, \dots, \tau_k]$ and $t_1, \dots, t_k$ are terms of types $\tau_1, \dots, \tau_k$, then $\alpha[t_1, \dots, t_k]$ is a formula, which is called atomic.

Formulas are constructed from atomic formulas by $\neg, \wedge, \forall$ applied to bound variables: if $A(a)$ is a formula the $\forall x A(x)$ is a formula if x is a new bound variable of the same type as a.

Alphabetical variant of an expression A is any result of renaming bound variables.

The height $h(\tau)$ of a type τ is the maximum nesting of [,]. If t is a term of type τ then $h(t) =_{def} h(\tau)$.

Substitution $A\binom{a}{t}$ of a term t for a free variable a of the same type into a formula or term A is defined as in [6], §20, i.e. bound variables are renamed to avoid collision, new occurrences of

$$\{x_1, \dots, x_k\}U(x_1, \dots, x_k)[t_1, \dots, t_k]$$

are converted into

$$U(b_1, \dots, b_k)\binom{b_1}{t_1\binom{a}{t}} \dots \binom{b_k}{t_k\binom{a}{t}} ,$$

and some η-conversions of $\{x_1,\ldots,x_k\}a[x_1,\ldots,x_k]$ into a are made.

If $V=\{x_1,\ldots,x_n\}A(x_1,\ldots,x_n)$ and $V_1,\ldots,V_n$ are terms of suitable types, then $V[V_1,\ldots,V_n]$ is defined to be

$$A(a_1,\ldots,a_n)\binom{a_1}{V_1}\binom{a_n}{V_n}$$

for suitable $a_1,\ldots,a_n$.

Derivable objects of the type theory with an the axiom of choice are *sequents* $\Gamma\to\Delta$ where Γ,Δ are finite sequences of formulas. The rules of inference are those of symple type theory with extensionality in §20, 21 of [6] including classical rules for logical connectives, structural rules, plus the following rules for ϵ-symbol:

$$(\epsilon)\ \frac{\Gamma\to\Delta, A(V)\qquad A(\epsilon x A(x)),\Gamma\to\Delta}{\Gamma\to\Delta}$$

where V is an abstract of a suitable type and ϵ-term $\epsilon x A(x)$ occurs in the conclusion $\Gamma\to\Delta$.

The *ϵ-extensionality rule* (*ext* ϵ) has a form:

$$\frac{A(a),\Gamma\to\Delta,B(a)\ \ B(a),\Gamma\to\Delta,A(a)\ \ \forall z(\epsilon x A(x)[z]\leftrightarrow\epsilon y B(y)[z]),\Gamma\to\Delta}{\Gamma\to\Delta}$$

where both terms $\epsilon x A(x),\epsilon y B(y)$ occur in the conclusion $\Gamma\to\Delta$ and a is a new variable of the same type as x.

Beginning here we restrict language like in §21 of [6] only to simplify notation: in particular there are no constants or function symbols, and types are $0,[0],[[0]]$ etc.

The standard extensionality rule is extended to account for ϵ-symbol:

$$(ext)\ \frac{V_1(a),\Gamma\to\Delta,V_2(a)\qquad V_2(a),\Gamma\to\Delta,V_1(a)}{\alpha[V_1],\Gamma\to\Delta,\alpha[V_2]}$$

where α is a free variable, predicate constant or an ϵ-term, a is a new variable, V_1,V_2 are arbitrary terms and types are coherent.

Recall also quantifier rules:

$$\frac{F(V),\Gamma\to\Delta}{\forall\psi F(\psi)\Gamma\to\Delta}\qquad\qquad\frac{\Gamma\to\Delta,F(a)}{\Gamma\to\Delta,\forall\psi F(\psi)}$$

Lemma 1 *ϵ-axiom, extensionality for ϵ-symbol, standard extensionality axiom and an axiom of choice are provable in simple type theory with extensionality and choice:*

$$A(V)\to A(\epsilon x A(x))$$

$$\forall x(A(x)\leftrightarrow B(x))\to\forall y(\epsilon x A(x)[y]\leftrightarrow\epsilon x B(x)[y])$$

$$\forall x(V_1(x)\leftrightarrow V_2(x))\to\forall\psi(\psi[V_1]\leftrightarrow\psi[V_2])$$

$$\forall x^\sigma\exists y^\tau A(x^\sigma,y^\tau)\to\exists z^{[\sigma,\tau]}\forall x^\sigma(\exists! y^\tau z^{[\sigma,\tau]}[x^\sigma,y^\tau]\wedge\forall y^\tau(z^{[\sigma,\tau]}[x^\sigma,y^\tau]\to A(x^\sigma,y^\tau)))$$

A *semi-formula (semi-term)* is an expression like a formula (term) except that it may contain free occurrences of bound variables.

2 The Cut-elimination Theorem

Definition 1 *(1) A* structure *(for simple type theory) is an ω-sequence $\mathsf{S}_0,\mathsf{S}_1,\ \ldots,\mathsf{S}_i,\ldots$ where S_0 is a non-empty set and S_{i+1} is a subset of $P(\mathsf{S}_i)$, the power set of S_i, $i=0,1,\ldots$*

A structure for simple type theory with choice is $\mathcal{S}=(\Phi,\mathsf{S}_0,\mathsf{S}_1,\ldots,\mathsf{S}_i,\ldots)$ where $\mathsf{S}_0,\mathsf{S}_1,\ldots,\mathsf{S}_i,\ldots$ is a structure for simple type theory and Φ is a choice function on $\bigcup_i P(\mathsf{S}_i)$:

for every non-empty set $\mathsf{S}\subset\mathsf{S}_i$ one has $\Phi(\mathsf{S})\in\mathsf{S}$

(2) An assignment ϕ *(from $\mathcal{S}$) is a mapping from all variables such that for every variable of type i ϕ assigns an element of S_i.*

Notation $\phi\binom{x}{\mathsf{S}}$ stands for the assignment which agrees with ϕ except at x where its value is S.

An interpretation $\Im$ is a pair consisting of a structure $\mathcal{S}$ and an assignment from $\mathcal{S}$.

(3) The interpretation (by $\Im = (\mathcal{S}, \phi)$) of semi-formulas and semi-terms e is defined in a standard way and denoted by $\phi(e)$.

$\phi(\alpha[W]) = \mathrm{T}$ *iff* $\phi(W) \in \phi(\alpha)$ *for a variable or ϵ-term α;*

$\phi(\forall x A(x)) = \mathrm{T}$ *iff for every ϕ' which agrees with ϕ except perhaps at x, $\phi'(A(x)) = \mathrm{T}$;*

$\phi(\{x\}A(x)) = \{\mathsf{S} \mid \mathsf{S} \in \mathsf{S}_i$ *and* $\phi\binom{x}{\mathsf{S}}(A(x)) = \mathrm{T}$ *where x is of type i*$\}$*;*

$\phi(\epsilon x A(x)) = \Phi(\ \phi(\{x\}A(x))\ )$*;*

$\phi(A \wedge B) = \mathrm{T}$ *iff* $\phi(A) = \mathrm{T}$ *and* $\phi(B) = \mathrm{T}$*;*

$\phi(\neg A) = \mathrm{T}$ *iff* $\phi(A) = \mathrm{F}$*.*

An interpretation is extended to sequents in a standard way.

A structure $\mathcal{S}$ is a *Henkin structure* if for every assignment ϕ and every abstract U^i of type i one has $\phi(U^i) \in S_i$.

Lemma 2 *Every sequent S provable in simple is type theory with extensionality and choice is true in any assignment from a Henkin structure with choice.*

Proof is standard: all rules are valid. $\dashv$

Definition 2 *A semivaluation* with extensionality and choice *is a partial assignment v of truth-values* T, F *to formulas, which satisfies standard conditions listed below. We say that v is* defined *for an ϵ-term e if v is defined for some formula containing e.*

1) $v(\neg A) = \mathrm{T}$ implies $v(A) = \mathrm{F}$; $v(\neg A) = \mathrm{F}$ implies $v(A) = \mathrm{T}$;

2) $v(A \wedge B) = \mathrm{T}$ implies $v(A) = \mathrm{T}$ and $v(B) = \mathrm{T}$; $v(A \wedge B) = \mathrm{F}$ implies $v(A) = \mathrm{F}$ or $v(B) = \mathrm{F}$;

3) $v(\forall x A(x)) = \mathrm{T}$ implies that $v(A(U)) = \mathrm{T}$ for every term U of the same type as x; $v(\forall x A(x)) = \mathrm{F}$ implies that $v(A(a)) = \mathrm{T}$ for some free variable a of the same type as x;

4) if A is an alphabetical variant of B, then $v(A) = v(B)$;

5) for any free variable or ϵ-term e of type > 1, if $v(e[U_1]) = \mathrm{T}$ and $v(e[U_2]) = \mathrm{F}$ then there is a free variable a of appropriate type such that either $v(U_1(a)) = \mathrm{T}$ and $v(U_2(a)) = \mathrm{F}$ or $v(U_1(a)) = \mathrm{F}$ and $v(U_2(a)) = \mathrm{T}$;

6) If v is defined for an ϵ-term $\epsilon x A(x)$, then either $v(A(\epsilon x A(x))) = \mathrm{T}$ or $v(A(U) = \mathrm{F}$ for all terms U of the same type as x.

7) If v is defined for ϵ-terms $\epsilon x A(x), \epsilon y B(y)$, then either $v(\forall z(\epsilon x A(x)[z] \leftrightarrow \epsilon y B(y)[z])) = \mathrm{T}$ or there is a free variable a of the same type as x such that either $v(A(a)) = \mathrm{T}$ and $v(B(a)) = \mathrm{F}$ or $v(A(a)) = \mathrm{F}$ and $v(B(a)) = \mathrm{T}$.

Lemma 3 *If a sequent S is not cut-free provable in the simple type theory with extensionality and choice, then there is a semivaluation v with extensionality and choice such that $v(S) = \mathrm{F}$.*

Proof is standard (cf. [6] Proposition 21.7): construct a canonical proof-search tree without cut by an exhaustive search, and take a non-closed branch. $\dashv$

Definition 3 *Given a semivaluation v with extensionality and choice, we define the structure* $(\mathsf{S}_0, \mathsf{S}_1, \ldots)$ *induced by v and relations $U^{n+1} < \mathsf{S}$ for abstracts U^{n+1} and $\mathsf{S} \subset \mathsf{S}_n$ exactly as in [6].*

S_0 is the set of all terms of type 0. $t_1 < t_2$ means that t_1 is identical to t_2.

$U^{n+1} < \mathsf{S}$ iff for every abstract U_0^n of type n and every $\mathsf{S}^n \in \mathsf{S}_n$, if $U_0^n < \mathsf{S}^n$ and $v(U^{n+1}[U_0^n]) = \mathrm{T}$ then $\mathsf{S}_n \in \mathsf{S}$, and if $U_0^n < \mathsf{S}^n$ and $v(U^{n+1}[U_0^n]) = \mathrm{F}$ then $\mathsf{S}_n \notin \mathsf{S}$.

$$\mathsf{S}_{n+1} =_{def} \{\mathsf{S} \mid \mathsf{S} \subset \mathsf{S}_n \ \text{ and there exists a } \ U^{n+1} \ \text{ such that } \ U^{n+1} < \mathsf{S}\}$$

We use the word "abstract" like in [6]: free variable of type 0 is an abstract, and to every free variable or an ϵ-term a we associate an abstract, also written a, namely a itself if a has type 0, and $\{x\}a[x]$ if a has type > 0.

For every free variable or ϵ-term α of type $n > 0$ set

$$\alpha^{(0)} =_{def} \{\mathsf{S}^{n-1} \mid (\exists U^{n-1} < \mathsf{S}^{n-1})\ v(\alpha[U^{n-1}]) = \mathrm{T}\}$$

If t is a term of type 0, $t^{(0)} =_{def} t$.

Lemma 4 *Let α be a free variable or an ϵ-term of type n. Then*

$$\alpha < \alpha^{(0)} \tag{1}$$

Proof. By induction on n exactly as in [6]. No special properties of ϵ-terms are used. $\dashv$

We say that a set $\mathsf{S} \subset \mathsf{S}_i$ *chooses* term $\epsilon x A(x)$ (under a semivaluation v) if v is defined for $\epsilon x A(x)$ and $\{x\}A(x) < \mathsf{S}$. Here x is of type i.

Note. By the clause 6) of the definition of semivaluation, if $\mathsf{S} \subset \mathsf{S}_i$ *chooses* a term $\epsilon x A(x)$, then $v(A(\epsilon x A(x))) = \mathrm{T}$ or $\mathsf{S} = \emptyset$. Indeed, if $v(A(\epsilon x A(x))) \neq \mathrm{T}$ then for any $\mathsf{S}^{n-1} \in \mathsf{S}$ there is an $U < \mathsf{S}^{n-1}$, and $\{x\}A(x) < \mathsf{S}$ and $v(A(U)) = \mathrm{F}$ imply $\mathsf{S}^{n-1} \notin \mathsf{S}$ as required.

Now define a function Φ (cf. Lemma 5 below) for $\mathsf{S} \subset \mathsf{S}_i$ by

$$\Phi(\mathsf{S}) = \begin{cases} \epsilon x A(x)^{(0)} & \text{if } \mathsf{S} \text{ chooses } \epsilon x A(x) \\ \text{some element of } \mathsf{S} & \text{if } \mathsf{S} \text{ chooses no } \epsilon\text{-term and } \mathsf{S} \neq 0 \\ \text{some element of } \mathsf{S}_i & \textit{otherwise} \end{cases}$$

We still have to prove that Φ is a function, since one and the same set can choose different ϵ-terms.

Lemma 5 *(a) If a set $\mathsf{S} \subset \mathsf{S}_n$ chooses $\epsilon x A(x)$ and $\epsilon y B(y)$ then $\epsilon x A(x)^{(0)} = \epsilon y B(y)^{(0)}$.*
(b) Φ is a choice function

Proof.(a) Note that v is defined for $\epsilon x A(x)$ and $\epsilon y B(y)$, and apply the clause 7) of the definition of a semi-valuation.

Case 1. $v(\forall z(\epsilon x A(x)[z] \leftrightarrow \epsilon y B(y)[z])) = \mathrm{T}$. Then by the clauses 3),1), 2) of the definition of a semi-valuation, v is defined for all formulas $\epsilon x A(x)[U^{n-1}]$, $\epsilon y B(y)[U^{n-1}]$ and every term U^{n-1} of type $n-1$, and

$$v(\epsilon x A(x)[U^{n-1}]) = v(\epsilon y B(y)[U^{n-1}]) \tag{2}$$

Take $\mathsf{S}^{n-1} \in \mathsf{S}_{n-1}$. One has

$$\mathsf{S}^{n-1} \in (\epsilon x A(x))^{(0)} \quad iff \quad v(\epsilon x A(x)[U^{n-1}]) = \mathrm{T} \text{ for some } \ U^{n-1} < \mathsf{S}^{n-1} \tag{3}$$

$$\mathsf{S}^{n-1} \in (\epsilon y B(y))^{(0)} \quad iff \quad v(\epsilon y B(y)[V^{n-1}]) = \mathrm{T} \text{ for some } \ V^{n-1} < \mathsf{S}^{n-1} \tag{4}$$

Assume for contradiction that $S^{n-1} \in (\epsilon x A(x))^{(0)},\ S^{n-1} \notin (\epsilon y B(y))^{(0)}$. By (3,4),

$$v(\epsilon x A(x)[U^{n-1}]) = \mathrm{T} \quad v(\epsilon y B(y)[V^{n-1}]) = \mathrm{T} \text{ for some } \ U^{n-1}, V^{n-1} < \mathsf{S}^{n-1}$$

By (2) this implies $v(\epsilon y B(y)[U^{n-1}]) = \mathrm{T}$, and by the clause 5) in the definition of the semivaluation, there is a variable a such than $v(U^{n-1}[a]) \neq v(V^{n-1}[a])$ although both are defined. This implies by $U^{n-1}, V^{n-1} < \mathsf{S}^{n-1}$ and $a < a^{(0)}$, that $a^{(0)} \in \mathsf{S},\ a^{(0)} \notin \mathsf{S}$, a contradiction.

Case 2. There is a free variable a of type n such that say $v(A(a)) = \mathrm{F}$ and $v(B(a)) = \mathrm{T}$. Since $a < a^{(0)}$ by (1), the relations $\{x\}A(x) < \mathsf{S},\ \{y\}B(y) < \mathsf{S}$ imply a contradiction: $a^{(0)} \in \mathsf{S},\ a^{(0)} \notin \mathsf{S}$.

(b) Let $\mathsf{S} \subset \mathsf{S}_n$. If S chooses some ϵ-term $e = \epsilon x A(x)$, then $\Phi(\mathsf{S}) = e^{(0)}$ is unique by (a), hence Φ is a function. If $v(e) = \mathrm{T}$ then since $\{x\}A(x) < \mathsf{S}$ and $e < e^{(0)}$, one has $e^{(0)} \in \mathsf{S}$ as required for a choice function. Otherwise by the clause 6 in the definition of a semivaluation, $v(A(U)) = \mathrm{F}$ for all U of type $n-1$. Then $\mathsf{S} = \emptyset$. $\dashv$.

We extend the relation $<$ to formulas and truth-values as follows:

$A < \mathrm{T}$ iff $v(A) \neq \mathrm{F}$;

$A < \mathrm{F}$ iff $v(A) \neq \mathrm{T}$;

Lemma 6 *Let $\mathcal{S}$ be a structure with extensionality and choice defined above and let ϕ be an assignment for $\mathcal{S}$. Then for any abstract or formula $U(\alpha_1, \ldots, \alpha_n)$ with all free variables among $\alpha_1, \ldots, \alpha_n$ and for any abstracts $U_1, \ldots, U_n$ of appropriate types, if $U_i < \phi(\alpha_i)$ for $i = 1, \ldots, n$ then*

$$U(U_1, \ldots, U_n) < \phi(U(\alpha_1, \ldots, \alpha_n)) \tag{5}$$

Proof. By induction on complexity of U as in the Proposition 21.11 of [6]. Consider only the new case

$$U(\alpha_1, \ldots, \alpha_n) = \epsilon x A(x, \alpha_1, \ldots, \alpha_n)$$

which is similar to the case (4) in [6]. If $v(U(U_1, \ldots, U_n))$ is undefined, the statement (5) is trivial. Otherwise set

$$Q =_{def} \{\mathsf{S} \mid \phi'(A(\beta, \alpha_1, \ldots, \alpha_n)) = \mathrm{T},\ \textit{where} \quad \phi' = \phi \binom{\beta}{\mathsf{S}}\}$$

i.e. $Q = \phi(\{x\}A(x, \alpha_1, \ldots, \alpha_n))$.

Then Q chooses $\epsilon x A(x, \alpha_1, \ldots, \alpha_n)$ since $\{x\}A(x, \alpha_1, \ldots, \alpha_n) < Q$ by the induction hypothesis. [We assume that ϵx contributes to the complexity more than $\{x\}$]. One has

$$\phi(\epsilon x A(x, \alpha_1, \ldots, \alpha_n)) = \Phi(Q)$$

and hence

$$\Phi(Q) = (\epsilon x A(U_1, \ldots, U_n))^{(0)}.$$

So (5) is (1). $\dashv$

Lemma 7 *$\mathcal{S}$ is a Henkin structure*

Proof. Exactly as in [6].

Theorem 1 *Cut-elimination holds for simple type theory with extensionality.*

Proof. As in [6]: apply Lemmas 2,3,5,6. $\dashv$

2.1 Second Order Logic with ϵ-symbol

Theorem 1 as it stands does not imply the Theorem 9.7 of [2]: cut-elimination for the system LK^2_ϵ of second order logic with ϵ-symbol. The language of LK^2_ϵ is obtained from the language of symple type theory described in the introduction by dropping all objects of type > 1 and some further less essential modifications. The rules of LK^2_ϵ are adaptations to this language of the rules described in the Introduction . The proof of a cut-elimination theorem for LK^2_ϵ is obtained from the proof of the Theorem 1 by the same adaptation. Important feature is the use of a third-order notion $\{x^1\}A(x^1)$ (where A is a formula of LK^2_ϵ) in the definition of the relation " a set $\mathsf{S} \subset \mathsf{S}_i$ *chooses* term $\epsilon x A(x)$". After this the whole proof goes through unchanged.

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