A proof of a theorem by Ikeda and Arai

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Abstract

K. Ikeda [2] gave a proof-theoretic demonstration of a normal form theorem for arithmetic derivations implying ω-consistency. Soon after T. Arai noticed a possibility to extend it to derivations in an arbitrary theory in the manner of [4]. We present here a short proof of these results which extends the proof in [4].

1 Irreducible Derivations

Consider a standard cut-free formulation of classical predicate logic where each rule has at most one side (=auxiliary) formula in each premise. For example $\rightarrow$-succedent has two forms:

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, A \rightarrow B} \qquad \vdash\rightarrow \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$$

Definition 1 *An inference (i.e. application of a rule) L is* redundant *with respect to a set $\mathcal{A}$ of formulas iff an antecedent side formula of L or a negation of a succedent side formula of L is implied by $\mathcal{A}$ in the predicate logic.*

For example the first of the rules $\vdash\rightarrow$ above is redundant if $A \vdash \mathcal{A}$, and the second of them is redundant if $\mathcal{A} \vdash \neg B$.

Definition 2 *d is a derivation* with proper variables *if a variable b can occur free in the premise of an inference L but not below only in the case, when b is the eigenvariable of L.*

Note that free variables of such a derivation are eigenvariables and free variables of its last sequent. The following statement is an easy consequence of the Theorem 1 in [4].

Theorem 1 *Every sequent provable in first order logic has a cut-free derivation with proper variables containing no inferences redundant with respect to $\emptyset$.*

Let $\mathcal{A}_0, \ldots, \mathcal{A}_{n+1}$ be finite sets of closed formulas.

Definition 3 *An inference L in a cut-free derivation of a sequent*

$$\mathcal{A}_0, \ldots, \mathcal{A}_n \vdash \mathcal{A}_{n+1} \tag{1}$$

*is n-*redundant *(w.r.t. $\mathcal{A}_0; \ldots; \mathcal{A}_n$) if for some i, $0 \leq i \leq n$, the main formula of L is traceable to $\mathcal{A}_{i+1}$ and L is redundant w.r.t. $\mathcal{A}_0, \ldots, \mathcal{A}_i$.*

We present a simple model-theoretic proof of the following statement due to T. Arai [1].

Theorem 2 *Every sequent (1) provable in first order logic has a cut-free derivation with proper variables containing no n-redundant inferences.*

Before presenting a proof, note the following

Corollary 1 *(after K. Ikeda [2]). PA is ω-consistent.*

Proof. Let $\mathcal{A}_0$ consist of closures of all axioms of PA different from induction, $\mathcal{A}_1$ consist of all closures of induction axioms

$$B[0]\&\forall y(B[y] \to B[y+1]) \to B[x] \tag{2}$$

and $\exists xA[x]$ be a sentence derivable in PA. By the Theorem 2 there is a logical derivation of the sequent

$$\mathcal{A}_0, \mathcal{A}_1 \vdash \exists xA[x] \tag{3}$$

with proper variables containing no redundant inferences. Consider inferences in such a derivation beginning from the very last one deriving the sequent (3). Since all closed instances of the formulas in $\mathcal{A}_1$ are logically derivable from $\mathcal{A}_0$, none of of the formulas in $\mathcal{A}_1$ is analysed before some eigenvariable appears. None of the inferences analysing formulas in $\mathcal{A}_0$ introduces eigenvariables, since the only quantifiers these formulas contain are initial $\forall$-quantifiers. Hence the final part of the derivation is as follows:

$$\cfrac{\begin{array}{c}\ldots\\ \mathcal{A}_0', \mathcal{A}_1' \vdash \Delta, A[t]\end{array}}{\mathcal{A}_0', \mathcal{A}_1' \vdash \Delta, \exists xA[x]}\ \vdash\exists$$
$$d$$
$$\mathcal{A}_0, \mathcal{A}_1 \vdash \exists xA[x]$$

where d consists of inferences traceable to $\mathcal{A}_0$ and structural inferences, and t is a constant term. If n is the value of t, then $\neg A[\bar{n}]$ is not derivable in PA (i.e. from $\mathcal{A}_0, \mathcal{A}_1$) since otherwise the inference $\vdash \exists$ would be redundant. □

Proof of the Theorem 2. Suppose that the sequent (1) (to be denoted by S) does not have a derivation satisfying our conditions. Consider its *canonical proof search tree* T_S respecting proper variable and irredundancy restrictions. Recall that such a tree is constructed bottom up by exhaustive search of all logical rules which can derive given sequent (cf. [3, Section 49], or [5]). Exhaustiveness of the search means (for usual construction) that any formula is eventually analysed with all possible side formulas. We substitute in the $\forall\vdash$- and $\vdash\exists$-inferences only terms containing proper variables, and we do not include n-redundant inferences. If for example an antecedent formula $A\&B$ is traceable to $\mathcal{A}_{i+1}$ and

$$\mathcal{A}_0, \ldots, \mathcal{A}_i \vdash A \qquad \text{is derivable,} \tag{4}$$

then A is not added to the antecedent.

As in the usual model-theoretic proof of the cut-elimination theorem, T_S cannot be closed (i.e. it is impossible that all branches end in axioms), since then it is a derivation satisfying our conditions. Hence it is possible to find (by König's Lemma) a non-closed branch W of T_S. Let us verify that it determines a countermodel for S with the universe U of all terms occurring in W. Denote by W_a, W_c the antecedent and succedent of W, i.e. the set of all formulas occurring in these parts of W. They satisfy the following properties, very similar to those used in the usual model-theoretic proof of the admissibility of cut. Here a formula F is called *redundant* w.r.t. W_a (W_c) and a formula $G \in W_a$ (resp. $G \in W_c$) if if G is traceable to $\mathcal{A}_{i+1}$ and $\mathcal{A}_0, \ldots, \mathcal{A}_i \vdash F$ is derivable (resp. $\mathcal{A}_0, \ldots, \mathcal{A}_i \vdash \neg F$ is derivable).

(a) W_a, W_c are disjoint w.r.t. atomic formulas

($\neg$). If $\neg F \in W_a$ ($\in W_c$) then either (F is redundant w.r.t. W_c and $\neg F$) or $F \in W_c$ (resp. F is redundant w.r.t. W_a and $\neg F$, or $F \in W_A$).

In the next clauses we say simply 'redundant' instead of 'redundant w.r.t. the suitable main formula'.

(&) If $(B\&C) \in W_a$ ($(B\&C) \in W_c$) then each of the formulas A, B is redundant or belongs to W_a (resp. at least one of the formulas A, B is redundant or belongs to W_c).

($\vdash\forall$) If $\forall xB[x] \in W_c$ then $B[t]$ for some t is redundant or belongs to W_c.

($\forall\vdash$) If $\forall xB[x] \in W_a$ then for every $t \in U$, $B[t]$ is redundant or belongs to W_c.

Denote by Φ the unique model with domain U given by the condition:

$$\Phi(E) = 1 \qquad iff \qquad E \in W_a \tag{5}$$

for atomic formulas E. To prove that $\Phi(S) = 0$ one proves

$$(C \in W_a \to \Phi(C) = 1) \& (C \in W_c \to \Phi(C) = 0) \tag{6}$$

by induction on (i, C) where C is traceable to $\mathcal{A}_i$. In fact most part of the induction requires only (complexity of) C as in the standard proof. Induction base is (a) above, and induction step uses properties $\&, \vdash \forall \vdash$ in a standard way. However the case of a redundant side formula requires special care.

Suppose for example that A is not added to the antecedent for an antecedent formula $A\&B$, since the latter is traceable to $\Phi(\mathcal{A}_{i+1})$ and (4) holds. Then by the induction hypothesis on i, we have

$$\Phi(\mathcal{A}_0, \ldots, \mathcal{A}_i) = 1$$

and hence by (4) $\Phi(A) = 1$. This concludes the proof of the theorem 2. □

References

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